Theoretical Seismology

Theorems in dynamic elasticity
Representation of the seismic source
Elastic wave propagation: body waves

(Based on: Aki & Richards, Quantitative Seismology)

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- Equation of motion
- Stress-strain relation for elastic medium

Strain tensor

Lagrangian approach: follows particle at specified position $\bar{x}$ and specified time t. (Alternative is Eulerian approach.)

Particle 1: at $\bar{x}$ with displacement $\bar{u}(\bar{x})$ moves to $\bar{x} + \bar{u}(\bar{x})$

Particle 2: at $\bar{x} + \delta\bar{x}$ with displacement $\bar{u}(\bar{x} + \delta\bar{x})$ moves to $\bar{x} + \delta\bar{x} + \bar{u}(\bar{x} + \delta\bar{x})$

Initial difference: $\delta\bar{x}$

New difference: $\delta\bar{x} + \delta\bar{u} = \bar{x} + \delta\bar{x} + \bar{u}(\bar{x} + \delta\bar{x}) - (\bar{x} + \bar{u}(\bar{x}))$

$$= \delta\bar{x} + \bar{u}(\bar{x} + \delta\bar{x}) - \bar{u}(\bar{x})$$

1st order Taylor expansion:

$$\bar{u}(\bar{x} + \delta\bar{x}) = \bar{u}(\bar{x}) + \delta\bar{u} \simeq \bar{u}(\bar{x}) + (\delta\bar{x} \cdot \nabla)\bar{u}$$

Strain tensor

where

$$\delta\bar{u} = (\delta\bar{x} \cdot \nabla)\bar{u}$$

$$\delta u_i = \frac{\partial u_i}{\partial x_j}\, \delta x_j$$

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\textit{rotation}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\textit{deformation}}$$

Strain tensor:

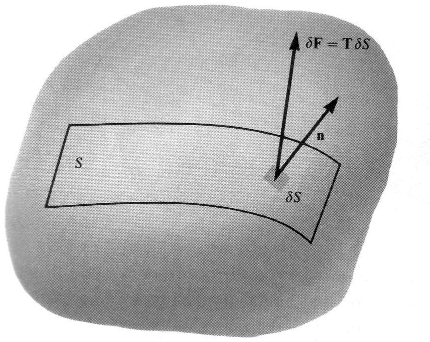
$$e_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Notation:

$$\frac{\partial u_i}{\partial x_j} = u_{i,j} \quad \textit{or} \quad \frac{\partial}{\partial x_j} = {}_{,j}$$

Surface forces

Traction $\bar{T}$ at a point across internal surface S with normal $\hat{n}$ (for a closed surface $\hat{n}$ is the outward directed normal):



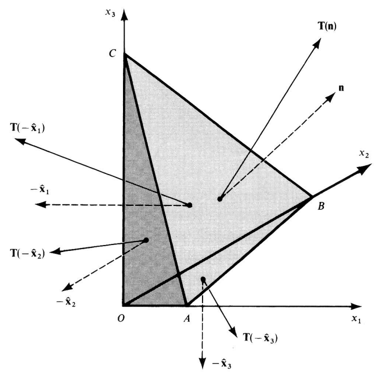
$$\bar{T}(\hat{n}) = lim_{\delta S \to 0} \frac{\delta \bar{F}}{\delta S}$$

We have, as $V \to 0$,

$$\int_S \bar{T} dS = \bar{0}$$

and therefore $\bar{T}(-\hat{n}) = -\bar{T}(\hat{n})$

Surface forces



also:

$$\bar{T}(\hat{n})S_{\hat{n}} + \bar{T}(-\hat{x}_1)S_{\hat{x}_1} + \bar{T}(-\hat{x}_2)S_{\hat{x}_2} + \bar{T}(-\hat{x}_3)S_{\hat{x}_3} = \bar{0}$$

with

$$S_{\hat{x}_1} = S_{\hat{n}} n_1 \qquad S_{\hat{x}_2} = S_{\hat{n}} n_2 \qquad S_{\hat{x}_3} = S_{\hat{n}} n_3$$

and volume tedrahedron $\to 0$:

$$\bar{T}(\hat{n}) = \bar{T}(\hat{x}_j) n_j \qquad (\textit{summation convention})$$

Surface forces

Again

$$\bar{T}(\hat{n}) = \bar{T}(\hat{x}_j) n_j$$

So

$$T_i(\hat{n}) = T_i(\hat{x}_j) n_j \equiv \tau_{ji} n_j$$

Furthermore, $\tau_{ji} = \tau_{ij}$
So

$$T_i(\hat{n}) = \tau_{ij} n_j$$

or

$$\bar{T}(\hat{n}) = \tau \hat{n}$$

Body forces

Body force works per unit volume (unit is $[N/m^3]$).

Point force $\bar{f}$:

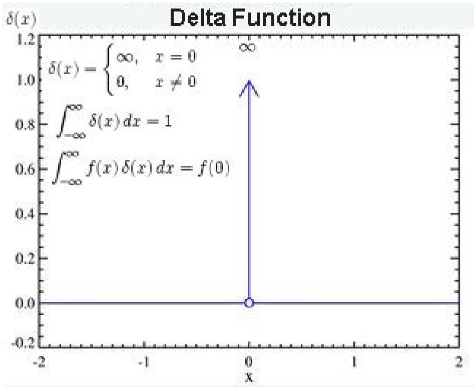
- in direction $\hat{x}_n$ axis
- at location $\bar{\xi}$
- at time τ

$$f_i(\bar{x}, t) = A\ \delta(\bar{x} - \bar{\xi})\ \delta(t - \tau)\ \delta_{in}$$

Dirac delta function

Definition delta function

$$\int_{-\infty}^{\infty} f(x)\ \delta(x)\ dx = f(0)$$



Surface area under delta function $= 1$ (width $\rightarrow 0$, height $\rightarrow \infty$).
Thus,

$$\int_{-\infty}^{\infty} f(x)\ \delta(x-a)\ dx = f(a)$$

Dirac delta function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\bar{x})\ \delta(\bar{x})\ dV = f(\bar{0})$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\bar{x})\ \delta(\bar{x}-\bar{\xi})\ dV = f(\bar{\xi})$$

Delta function is derivative of Heaviside step function, $H(t)$:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Kronecker delta

Definition

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Body forces

Body force works per unit volume (unit is $[N/m^3]$).

Point force $\bar{f}$:

- in direction $\hat{x}_n$ axis
- at location $\bar{\xi}$
- at time τ

$$f_i(\bar{x}, t) = A\ \delta(\bar{x} - \bar{\xi})\ \delta(t - \tau)\ \delta_{in}$$

Scalar A has dimension of impulse, unit is $[Ns]$.

Equation of motion

Rate of change of momentum of particles in V

$$\frac{\partial}{\partial t}\int_V \rho \frac{\partial \bar{u}}{\partial t}\, dV = \int_V \bar{f}\, dV + \int_S \bar{T}(\hat{n}) dS$$

In Lagrangian description ρdV constant:

$$\int_V \rho \frac{\partial^2 \bar{u}}{\partial t^2}\, dV = \int_V \bar{f}\, dV + \int_S \bar{T}(\hat{n}) dS$$

We have

$$\int_S T_i dS = \int_S \tau_{ij} n_j dS$$

With the divergence theorem

$$\int_S T_i dS = \int_V \frac{\partial \tau_{ij}}{\partial x_j} dV$$

Thus,

$$\int_V \left(\rho \frac{\partial^2 u_i}{\partial t^2} - f_i - \frac{\partial \tau_{ij}}{\partial x_j}\right) dV = 0$$

Equation of motion

Again:

$$\int_V \left(\rho \frac{\partial^2 u_i}{\partial t^2} - f_i - \frac{\partial \tau_{ij}}{\partial x_j}\right) dV = 0$$

or

Equation of motion:

$$\rho \ddot{u}_i = f_i + \tau_{ij,j}$$

Stress-strain

Elastic medium: medium with natural state with zero stress and strain to which it will return when forces are removed.

Generalization of Hooke's law:

$$\tau_{ij} = c_{ijpq} e_{pq}$$

c_{ijpq} is 4th order elasticity tensor with 81 components

Symmetries:

- $\tau_{ij} = \tau_{ji} \longrightarrow c_{ijpq} = c_{jipq}$
- $e_{pq} = e_{qp} \longrightarrow c_{ijpq} = c_{ijqp}$

Stress-strain

Thermodynamics: strain-energy function
$W = \frac{1}{2} c_{ijpq} e_{ij} e_{pq} = \frac{1}{2} \tau_{ij} e_{ij}$

$$W = \frac{1}{2} \tau_{ij} e_{ij} = \frac{1}{2} c_{ijpq} e_{pq} e_{ij}$$

$$W = \frac{1}{2} \tau_{pq} e_{pq} = \frac{1}{2} c_{pqij} e_{ij} e_{pq}$$

- $c_{ijpq} = c_{pqij}$

Due to symmetries c_{ijpq} has 21 independent components

Stress-strain

Isotropy:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with Lamé's constants λ and μ.

It can be shown that:

$$\tau_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

with dilation θ:

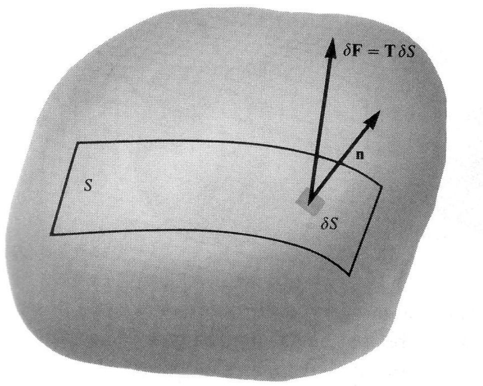
$$\theta = e_{11} + e_{22} + e_{33} = e_{ii}$$

- Green's function
- Reciprocity
- Representation theorems

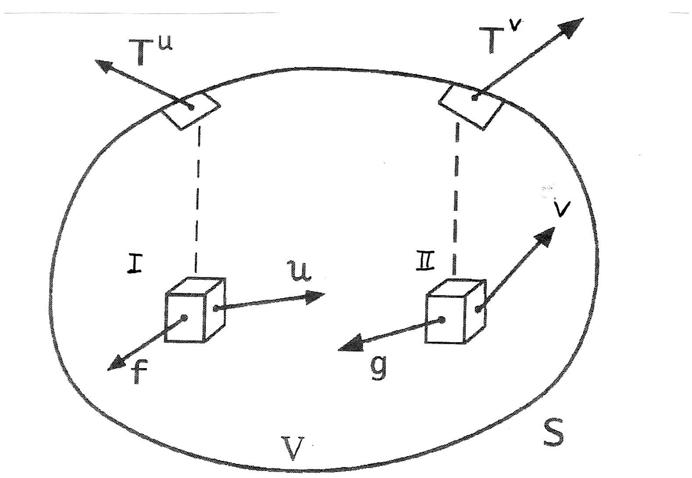
Uniqueness

$\bar{u}(\bar{x}, t)$ is uniquely determined after t_0 throughout V with surface S given:

- $\bar{u}(\bar{x}, t_0)$ and $\dot{\bar{u}}(\bar{x}, t_0)$
- $\bar{f}(\bar{x}, t \geq t_0)$
- $\bar{T}(\bar{x}, t \geq t_0)$ over $S_1 \leq S$
- $\bar{u}(\bar{x}, t \geq t_0)$ over S - S_1



Betti's theorem

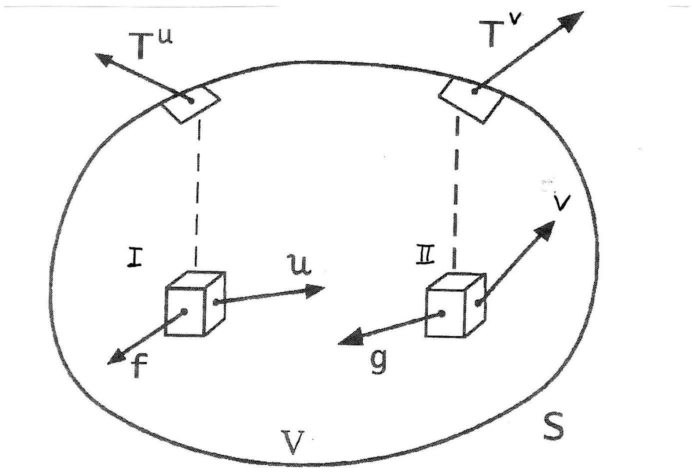


I:

$$\left.\begin{array}{c}\bar{f}(\bar{x}, t) \\ \bar{T}^{\bar{u}}(\bar{x}, t)\end{array}\right\} \rightarrow \bar{u}(\bar{x}, t)$$

$$\int_V (f_i - \rho \ddot{u}_i) dV + \int_S T_i^{\bar{u}} dS = 0$$

Betti's theorem



II:

$$\left.\begin{array}{c}\bar{g}(\bar{x},t)\\ \bar{T}^{\bar{v}}(\bar{x},t)\end{array}\right\} \rightarrow \bar{v}(\bar{x},t)$$

$$\int_V (g_i - \rho \ddot{v}_i) dV + \int_S T_i^{\bar{v}} dS = 0$$

Betti's theorem

Take scalar product with $\bar{v}$ and $\bar{u}$ respectively:

$$\int_V (f_i - \rho \ddot{u}_i) v_i \, dV + \int_S T_i^{\bar{u}} v_i \, dS = \int_V (g_i - \rho \ddot{v}_i) u_i \, dV + \int_S T_i^{\bar{v}} u_i \, dS$$

This is Betti's theorem.

It gives the reciprocal relation between the displacements corresponding to 2 systems of forces and tractions acting on the same volume.

Betti's theorem with causality

Betti's theorem integrated over time:

$$\int_{-\infty}^{\infty} dt \int_V \rho(\ddot{u}_i v_i - u_i \ddot{v}_i) dV =$$

$$\int_{-\infty}^{\infty} dt \int_V (v_i f_i - u_i g_i) dV + \int_{-\infty}^{\infty} dt \int_S (v_i T_i^{\bar{u}} - u_i T_i^{\bar{v}}) dS$$

Evaluate system I at $t_I = t$, and system II at $t_{II} = \tau - t$
l.h.s. becomes (change order of integration):

$$\int_V dV \int_{-\infty}^{\infty} \rho[\ddot{u}_i(t) v_i(\tau - t) - u_i(t) \ddot{v}_i(\tau - t)] dt$$

Causality (i.e. medium at rest until distorted):

$$\left. \begin{array}{l} u_i = \dot{u}_i = 0 \\ v_i = \dot{v}_i = 0 \end{array} \right\} t \leq 0$$

Betti's theorem with causality

$$\int_0^{\tau} \rho \frac{\partial}{\partial t} [\dot{u}_i(t) v_i(\tau - t) + \dot{v}_i(\tau - t) u_i(t)] dt =$$

$$\rho[\dot{u}_i(\tau) v_i(0) + \dot{v}_i(0) u_i(\tau) - \dot{u}_i(0) v_i(\tau) - \dot{v}_i(\tau) u_i(0)] = 0$$

With causality l.h.s. is zero, thus for r.h.s.

$$\int_{-\infty}^{\infty} dt \int_V (u_i g_i - v_i f_i) dV = \int_{-\infty}^{\infty} dt \int_S (v_i T_i^{\bar{u}} - u_i T_i^{\bar{v}}) dS$$

or

$$\int_{-\infty}^{\infty} dt \int_V [\bar{u}(\bar{x}, t) \cdot \bar{g}(\bar{x}, \tau - t) - \bar{v}(\bar{x}, \tau - t) \cdot \bar{f}(\bar{x}, t)] dV =$$

$$\int_{-\infty}^{\infty} dt \int_S [\bar{v}(\bar{x}, \tau - t) \cdot \bar{T}^{\bar{u}}(\bar{x}, t) - \bar{u}(\bar{x}, t) \cdot \bar{T}^{\bar{v}}(\bar{x}, \tau - t)] dS$$

Betti's theorem with causality

Betti's theorem (with causality):

$$\int_{-\infty}^{\infty} dt \int_V [\bar{u}(\bar{x}, t) \cdot \bar{g}(\bar{x}, \tau - t) - \bar{v}(\bar{x}, \tau - t) \cdot \bar{f}(\bar{x}, t)] dV =$$

$$\int_{-\infty}^{\infty} dt \int_S [\bar{v}(\bar{x}, \tau - t) \cdot \bar{T}^{\bar{u}}(\bar{x}, t) - \bar{u}(\bar{x}, t) \cdot \bar{T}^{\bar{v}}(\bar{x}, \tau - t)] dS$$

Important, because it allows the representation of the displacement due to a system of forces by those produced by a different system (given that causality is satisfied).
Displacements due to a complicated system of forces can be represented in terms of those produced by a simpler one.

Green's function

The displacement field due to a unit impulse in space and time is the elastodynamic Green's function.

Green's function:

The ith component of displacement at location $\bar{x}$ and time t due to a unit impulse acting in the $\hat{x}_n$-direction at location $\bar{\xi}$ and time τ is denoted by the Green's function:

$$G_{in}(\bar{x}, t; \bar{\xi}, \tau)$$

Or

$$u_i(\bar{x}, t) = G_{in}(\bar{x}, t; \bar{\xi}, \tau)$$

due to

$$f_i(\bar{x}, t) = \delta_{in}\delta(\bar{x} - \bar{\xi})\delta(t - \tau)$$

Green's function

Equation of motion:

$$\rho \ddot{u}_i = f_i + \tau_{ij,j} = f_i + (c_{ijkl} u_{k,l})_{,j} \qquad (\textit{see problem } 2.1)$$

for $u_i(\bar{x}, t) = G_{in}(\bar{x}, t; \bar{\xi}, \tau)$ due to a unit impulse at $(\bar{\xi}, \tau)$ in the $\hat{x}_n$-direction becomes

$$\rho \frac{\partial^2}{\partial t^2} G_{in}(\bar{x}, t; \bar{\xi}, \tau) = \delta_{in} \delta(\bar{x} - \bar{\xi}) \delta(t - \tau) + \frac{\partial}{\partial x_j} \left(c_{ijkl} \frac{\partial}{\partial x_l} G_{kn}(\bar{x}, t; \bar{\xi}, \tau) \right)$$

Reciprocity

Initial conditions:

$$\left.\begin{array}{r} \mathbf{G}(\bar{x}, t; \bar{\xi}, \tau) = 0 \\ \frac{\partial}{\partial t} \mathbf{G}(\bar{x}, t; \bar{\xi}, \tau) = 0 \end{array}\right\} \text{for} \quad t \leq \tau \quad \text{and} \quad \bar{x} \neq \bar{\xi}$$

and

$$\rho \frac{\partial^2}{\partial t^2} G_{in}(\bar{x}, t; \bar{\xi}, \tau) = \delta_{in} \delta(\bar{x} - \bar{\xi}) \delta(t - \tau) + \frac{\partial}{\partial x_j} \left(c_{ijkl} \frac{\partial}{\partial x_l} G_{kn}(\bar{x}, t; \bar{\xi}, \tau) \right)$$

(a) Boundary conditions independent of time (i.e. rigid boundary):
Time dependence only through $t - \tau$:

Time reciprocity

$$\mathbf{G}(\bar{x}, t; \bar{\xi}, \tau) = \mathbf{G}(\bar{x}, t - \tau; \bar{\xi}, 0) = \mathbf{G}(\bar{x}, -\tau; \bar{\xi}, -t)$$

Reciprocity

(b) Traction free boundary condition on S and:
for system I:

$$f_i(\bar{x}, t) = \delta_{im}\delta(\bar{x} - \bar{\xi}_1)\delta(t - \tau_1) \quad \rightarrow u_i(\bar{x}, t) = G_{im}(\bar{x}, t; \bar{\xi}_1, \tau_1)$$

for system II:

$$g_i(\bar{x}, t) = \delta_{in}\delta(\bar{x} - \bar{\xi}_2)\delta(t + \tau_2) \quad \rightarrow v_i(\bar{x}, t) = G_{in}(\bar{x}, t; \bar{\xi}_2, -\tau_2)$$

Then with Betti's theorem for causal conditions

$$\int_{-\infty}^{\infty} dt \int_V G_{im}(\bar{x}, t; \bar{\xi}_1, \tau_1)\delta_{in}\delta(\bar{x} - \bar{\xi}_2)\delta(\tau - t + \tau_2)dV =$$

$$\int_{-\infty}^{\infty} dt \int_V G_{in}(\bar{x}, \tau - t; \bar{\xi}_2, -\tau_2)\delta_{im}\delta(\bar{x} - \bar{\xi}_1)\delta(t - \tau_1)dV$$

or

$$G_{nm}(\bar{\xi}_2, \tau + \tau_2; \bar{\xi}_1, \tau_1) = G_{mn}(\bar{\xi}_1, \tau - \tau_1; \bar{\xi}_2, -\tau_2)$$

Reciprocity

$$G_{nm}(\bar{\xi}_2, \tau + \tau_2; \bar{\xi}_1, \tau_1) = G_{mn}(\bar{\xi}_1, \tau - \tau_1; \bar{\xi}_2, -\tau_2)$$

- With $\tau_1 = \tau_2 = 0$, we have

Spatial reciprocity

$$G_{nm}(\bar{\xi}_2, \tau; \bar{\xi}_1, 0) = G_{mn}(\bar{\xi}_1, \tau; \bar{\xi}_2, 0)$$

- With $\tau = 0$, we have

$$G_{nm}(\bar{\xi}_2, \tau_2; \bar{\xi}_1, \tau_1) = G_{mn}(\bar{\xi}_1, -\tau_1; \bar{\xi}_2, -\tau_2)$$

$\rightarrow$ space-time reciprocity

Representation theorem

We had Betti's theorem with causality:

$$\int_{-\infty}^{\infty} dt \int_V (u_i g_i - v_i f_i) dV = \int_{-\infty}^{\infty} dt \int_S (v_i T_i^{\bar{u}} - u_i T_i^{\bar{v}}) dS$$

With

$$g_i(\bar{x}, t) = \delta_{in}\delta(\bar{x} - \bar{\xi})\delta(t) \quad giving \quad v_i(\bar{x}, t) = G_{in}(\bar{x}, t; \bar{\xi}, 0)$$

and

$$T_i^{\bar{v}} = \tau_{ij} n_j = c_{ijkl} v_{k,l} n_j = c_{ijkl} G_{kn,l} n_j$$

we get

$$\int_{-\infty}^{\infty} dt \int_V [u_i(\bar{x}, t)\delta_{in}\delta(\bar{x}-\bar{\xi})\delta(\tau-t) - G_{in}(\bar{x}, \tau-t; \bar{\xi}, 0) f_i(\bar{x}, t)] dV =$$

$$\int_{-\infty}^{\infty} dt \int_S [G_{in}(\bar{x}, \tau-t; \bar{\xi}, 0) T_i^{\bar{u}}(\bar{x}, t) - u_i(\bar{x}, t) c_{ijkl} G_{kn,l}(\bar{x}, \tau-t, \bar{\xi}, 0) n_j] dS$$

Representation theorem

With

$$\int_{-\infty}^{\infty} dt \int_V [u_i(\bar{x}, t)\delta_{in}\delta(\bar{x} - \bar{\xi})\delta(\tau - t)] dV = u_n(\bar{\xi}, \tau)$$

we find

$$u_n(\bar{\xi}, \tau) = \int_{-\infty}^{\infty} dt \int_V G_{in} f_i dV + \int_{-\infty}^{\infty} dt \int_S [G_{in} T_i^{\bar{u}} - u_i n_j c_{ijkl} G_{kn,l}] dS$$

Change of variables:
$\bar{x} \rightarrow \bar{\xi}$, $t \rightarrow \tau$, $\bar{\xi} \rightarrow \bar{x}$, $\tau \rightarrow t$:

$$u_n(\bar{x}, t) = \int_{-\infty}^{\infty} d\tau \int_V G_{in}(\bar{\xi}, t - \tau; \bar{x}, 0) f_i(\bar{\xi}, \tau) dV(\bar{\xi}) +$$

$$\int_{-\infty}^{\infty} d\tau \int_S [G_{in}(\bar{\xi}, t-\tau; \bar{x}, 0) T_i^{\bar{u}}(\bar{\xi}, \tau) - u_i(\bar{\xi}, \tau) n_j c_{ijkl} G_{kn,l}(\bar{\xi}, t-\tau; \bar{x}, 0)] dS(\bar{\xi})$$

Representation theorem

$$
\begin{aligned}
u_n(\bar{x},t) &= \int_{-\infty}^{\infty} d\tau \int_V G_{in}(\bar{\xi},t-\tau;\bar{x},0) f_i(\bar{\xi},\tau) dV(\bar{\xi}) \\
&+ \int_{-\infty}^{\infty} d\tau \int_S [G_{in}(\bar{\xi},t-\tau;\bar{x},0) T_i^{\bar{u}}(\bar{\xi},\tau) \\
&\qquad - u_i(\bar{\xi},\tau) n_j c_{ijkl} G_{kn,l}(\bar{\xi},t-\tau;\bar{x},0)] dS(\bar{\xi})
\end{aligned}
$$

The integrals involve a Green's function with source at $\bar{x}$ and observation point $\bar{\xi}$. Apply spatial reciprocity:
$G_{in}(\bar{\xi},t-\tau;\bar{x},0) = G_{ni}(\bar{x},t-\tau;\bar{\xi},0)$
Representation theorem:

$$
\begin{aligned}
u_n(\bar{x},t) &= \int_{-\infty}^{\infty} d\tau \int_V G_{ni}(\bar{x},t-\tau;\bar{\xi},0) f_i(\bar{\xi},\tau) dV(\bar{\xi}) \\
&+ \int_{-\infty}^{\infty} d\tau \int_S [G_{ni}(\bar{x},t-\tau;\bar{\xi},0) T_i^{\bar{u}}(\bar{\xi},\tau) \\
&\qquad - u_i(\bar{\xi},\tau) n_j c_{ijkl} G_{nk,l}(\bar{x},t-\tau;\bar{\xi},0)] dS(\bar{\xi})
\end{aligned}
$$

Representation theorem with boundary conditions

(a) Rigid boundary condition:
$v_i(\bar{\xi},t-\tau) = G_{in}^{rigid}(\bar{\xi},t-\tau;\bar{x},0) = 0$ for $\bar{\xi}$ on S:

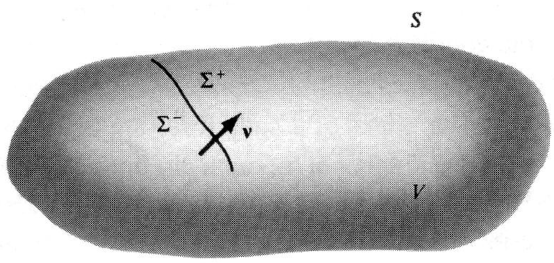
$$
\begin{aligned}
u_n(\bar{x},t) &= \int_{-\infty}^{\infty} d\tau \int_V f_i(\bar{\xi},\tau) G_{ni}^{rigid}(\bar{x},t-\tau;\bar{\xi},0) dV(\bar{\xi}) \\
&- \int_{-\infty}^{\infty} d\tau \int_S u_i(\bar{\xi},\tau)\; c_{ijkl} n_j G_{nk,l}^{rigid}(\bar{x},t-\tau;\bar{\xi},0) dS(\bar{\xi})
\end{aligned}
$$

(b) Traction free boundary condition:
$T_i^{\bar{v}}(\bar{\xi},t-\tau) = c_{ijkl} n_j G_{kn,l}^{free}(\bar{\xi},t-\tau;\bar{x},0) = 0$ for $\bar{\xi}$ on S:

$$
\begin{aligned}
u_n(\bar{x},t) &= \int_{-\infty}^{\infty} d\tau \int_V f_i(\bar{\xi},\tau) G_{ni}^{free}(\bar{x},t-\tau;\bar{\xi},0) dV(\bar{\xi}) \\
&+ \int_{-\infty}^{\infty} d\tau \int_S G_{ni}^{free}(\bar{x},t-\tau;\bar{\xi},0) T_i^{\bar{u}}(\bar{\xi},\tau) dS(\bar{\xi})
\end{aligned}
$$

Representation of seismic sources

Fault slip: body force representation



Slip across Σ → discontinuous displacement at Σ → equation of motion not valid at Σ. Equation of motion is valid at all other points → apply representation theorem:

$$u_n(\bar{x}, t) = \int_{-\infty}^{\infty} d\tau \int_V G_{ni} f_i dV(\bar{\xi}) + \int_{-\infty}^{\infty} d\tau \int_{\Sigma+S} [G_{ni} T_i^{\bar{u}} - u_i n_j c_{ijkl} G_{nk,l}] dS(\bar{\xi})$$

or with $\hat{n} \rightarrow \hat{\nu}$ (fault/surface normal), $k \rightarrow p$, $l \rightarrow q$:

$$u_n(\bar{x}, t) = \int_{-\infty}^{\infty} d\tau \int_V G_{ni} f_i dV(\bar{\xi}) + \int_{-\infty}^{\infty} d\tau \int_{\Sigma+S} [G_{ni} T_i^{\bar{u}} - u_i \nu_j c_{ijpq} G_{np,q}] dS(\bar{\xi})$$

Fault slip: body force representation

With $\bar{f} = \bar{0}$ and homogeneous boundary conditions on S $(\int_S G_{in} T_i^{\bar{u}} - u_i n_j c_{ijkl} G_{kn,l} dS = 0)$, and fault surface integration using slip $\bar{u}$ at fault :

$$u_n(\bar{x}, t) = -\int_{-\infty}^{\infty} d\tau \left[\int_{\Sigma^+} u_i^+(\bar{\xi}, \tau) \nu_j^+ c_{ijpq} \frac{\partial}{\partial \xi_q} G_{np}(\bar{x}, t-\tau; \bar{\xi}, 0) d\Sigma \right.$$
$$\left. + \int_{\Sigma^-} u_i^-(\bar{\xi}, \tau) \nu_j^- c_{ijpq} \frac{\partial}{\partial \xi_q} G_{np}(\bar{x}, t-\tau; \bar{\xi}, 0) d\Sigma \right]$$

Define fault normal $\nu_j = \nu_j^+ = -\nu_j^-$ so that:

$$-\left[\int_{\Sigma^+} u_i^+ \nu_j^+ d\Sigma + \int_{\Sigma^-} u_i^- \nu_j^- d\Sigma \right] = -\int_{\Sigma} (u_i^+ - u_i^-) \nu_j d\Sigma = \int_{\Sigma} \Delta u_i \nu_j d\Sigma$$

Then

$$u_n(\bar{x}, t) = \int_{-\infty}^{\infty} d\tau \int_{\Sigma} \Delta u_i(\bar{\xi}, \tau) \nu_j c_{ijpq} \frac{\partial}{\partial \xi_q} G_{np}(\bar{x}, t-\tau; \bar{\xi}, 0) d\Sigma$$

Fault slip: body force representation

Displacement $u_n(\bar{x}, t)$ due to slip $\Delta\bar{u}$ across Σ with normal $\bar{\nu}$:

$$u_n(\bar{x}, t) = \int_{-\infty}^{\infty} d\tau \int_{\Sigma} \Delta u_i(\bar{\xi}, \tau) \nu_j c_{ijpq} \frac{\partial}{\partial \xi_q} G_{np}(\bar{x}, t-\tau; \bar{\xi}, 0) d\Sigma$$

To represent fault slip as body force a volume integral is needed. Use 'trick':

$$G_{np}(\bar{x}, t-\tau; \bar{\xi}, 0) = \int_V G_{np}(\bar{x}, t-\tau; \bar{\eta}, 0) \delta(\bar{\eta} - \bar{\xi}) dV(\bar{\eta})$$

$$\begin{aligned} \frac{\partial}{\partial \xi_q} G_{np}(\bar{x}, t-\tau; \bar{\xi}, 0) &= \int_V G_{np}(\bar{x}, t-\tau; \bar{\eta}, 0) \frac{\partial}{\partial \xi_q} \delta(\bar{\eta} - \bar{\xi}) dV(\bar{\eta}) \\ &= -\int_V G_{np}(\bar{x}, t-\tau; \bar{\eta}, 0) \frac{\partial}{\partial \eta_q} \delta(\bar{\eta} - \bar{\xi}) dV(\bar{\eta}) \end{aligned}$$

Fault slip: body force representation

Therefore

$$u_n(\bar{x},t) = \int_{-\infty}^{\infty} d\tau \int_{\Sigma} \Delta u_i(\bar{\xi},\tau)\nu_j c_{ijpq} \frac{\partial}{\partial \xi_q} G_{np}(\bar{x},t-\tau;\bar{\xi},0)d\Sigma =$$

$$\int_{-\infty}^{\infty} d\tau \int_{V} \left[- \int_{\Sigma} \Delta u_i(\bar{\xi},\tau)\nu_j c_{ijpq} \frac{\partial}{\partial \eta_q} \delta(\bar{\eta}-\bar{\xi})d\Sigma(\bar{\xi}) \right] G_{np}(\bar{x},t-\tau;\bar{\eta},0)dV(\bar{\eta})$$

Fault slip $\Delta\bar{u}$ on Σ is represented by body-force equivalent $\bar{f}^{\Delta\bar{u}}$:

$$f_p^{\Delta\bar{u}}(\bar{\eta},\tau) = - \int_{\Sigma} \Delta u_i(\bar{\xi},\tau)\nu_j c_{ijpq} \frac{\partial}{\partial \eta_q} \delta(\bar{\eta}-\bar{\xi})d\Sigma(\bar{\xi})$$

to give 'predicted' displacements

$$u_n(\bar{x},t) = \int_{-\infty}^{\infty} d\tau \int_{V} f_p^{\Delta\bar{u}}(\bar{\eta},\tau) G_{np}(\bar{x},t-\tau;\eta,0)dV(\bar{\eta})$$

Convolution

Convolution of $f(t)$ and $g(t)$ is defined as:

$$h(t) = f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

For causal functions f and g $\quad (f(t) = g(t) = 0 \;\; t < 0)$:

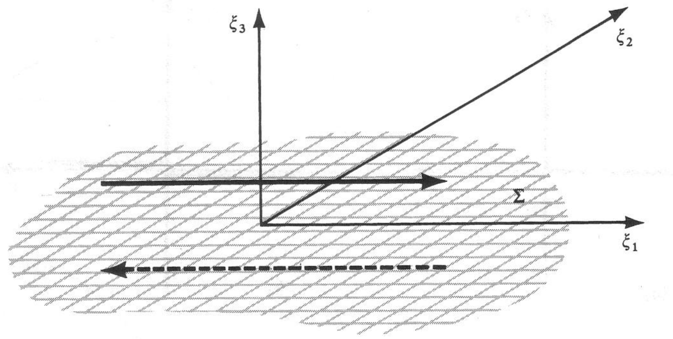
$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$$

(Note: if $f(t) = \delta(t)$: $\quad h(t) = f * g = g(t)$)

Application to body force distribution $\bar{f}(\bar{x},t)$:

$$\begin{aligned} u_n(\bar{x},t) &= \int_{-\infty}^{\infty} d\tau \int_{V} f_p(\bar{\eta},\tau) G_{np}(\bar{x},t-\tau;\bar{\eta},0)dV(\bar{\eta}) \\ &= \int_V f_p * G_{np} \; dV \end{aligned}$$

Example: Slip on a fault



Fault plane Σ in plane $\xi_3 = 0$
Fault normal $\hat{\nu} = (0, 0, 1)$
Fault slip $\Delta\bar{u} = (\Delta u_1, 0, 0)$
Body-force equivalent

$$f_p(\bar{\eta}, \tau) = -\int_\Sigma \Delta u_i(\bar{\xi}, \tau)\nu_j c_{ijpq} \frac{\partial}{\partial \eta_q}\delta(\bar{\eta} - \bar{\xi})d\Sigma(\bar{\xi})$$

becomes

$$f_p(\bar{\eta}, \tau) = -\int_\Sigma \Delta u_1(\bar{\xi}, \tau) c_{13pq} \frac{\partial}{\partial \eta_q}\delta(\bar{\eta} - \bar{\xi})d\xi_1 d\xi_2$$

Example: Slip on a fault

For isotropic medium:

$$c_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$$

$c_{13pq} = 0$ except for $c_{1313} = \mu$ and $c_{1331} = \mu$

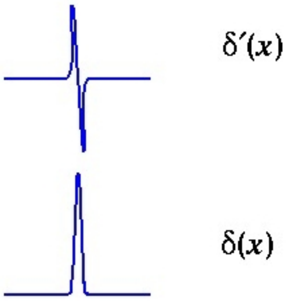
$$\begin{aligned}
f_1(\bar{\eta}, \tau) &= -\int_\Sigma \Delta u_1(\bar{\xi}, \tau)\mu(\bar{\xi}) \frac{\partial}{\partial \eta_3}\delta(\bar{\eta} - \bar{\xi})d\xi_1 d\xi_2 \\
f_2(\bar{\eta}, \tau) &= 0 \\
f_3(\bar{\eta}, \tau) &= -\int_\Sigma \Delta u_1(\bar{\xi}, \tau)\mu(\bar{\xi}) \frac{\partial}{\partial \eta_1}\delta(\bar{\eta} - \bar{\xi})d\xi_1 d\xi_2
\end{aligned}$$

$\delta(\bar{\eta} - \bar{\xi}) = \delta(\eta_1 - \xi_1)\delta(\eta_2 - \xi_2)\delta(\eta_3 - \xi_3)$ with $\xi_3 = 0$ on Σ

First consider f_1:
$f_1(\bar{\eta}, \tau) = -\mu(\bar{\eta})\Delta u_1(\bar{\eta}, \tau)\frac{\partial}{\partial \eta_3}\delta(\eta_3)$ on Σ, elsewhere $f_1 = 0$

Example: Slip on a fault

Using



$$\bar{M} = \bar{r} \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \eta_1 & \eta_2 & \eta_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$$

it can be shown that $f_1(\bar{\eta}, \tau) = -\mu(\bar{\eta})\Delta u_1(\bar{\eta}, \tau)\frac{\partial}{\partial \eta_3}\delta(\eta_3)$ (only on Σ!) represents a force couple ($\frac{\partial}{\partial \eta_3}\delta(\eta_3)$) with moment M_2:

$$\begin{aligned} M_2 = \int_V \eta_3 f_1 dV &= \int_V -\eta_3 \mu \Delta u_1 \delta'(\eta_3) d\eta_1 d\eta_2 d\eta_3 \\ &= \int_\Sigma \mu \Delta u_1 d\Sigma = \mu \overline{\Delta u_1} \Sigma \end{aligned}$$

with $\overline{\Delta u_1}$ the average slip and Σ the fault area.

Example: Slip on a fault

Next consider $f_3(\bar{\eta}, \tau) = -\int_\Sigma \mu(\bar{\xi})\Delta u_1(\bar{\xi}, \tau)\frac{\partial}{\partial \eta_1}\delta(\bar{\eta} - \bar{\xi})d\xi_1 d\xi_2$ ($\neq 0$ on Σ, elsewhere $f_3 = 0$).
f_3 does not represent a force couple:

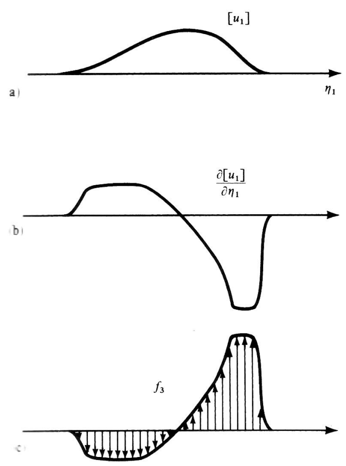
$$\begin{aligned} f_3(\bar{\eta}, \tau) &= -\int_\Sigma \mu(\bar{\xi})\Delta u_1(\bar{\xi}, \tau)\frac{\partial}{\partial \eta_1}[\delta(\eta_1 - \xi_1)]\delta(\eta_2 - \xi_2)\delta(\eta_3)d\xi_1 d\xi_2 \\ &= -\frac{\partial}{\partial \eta_1}\left[\mu(\bar{\eta})\Delta u_1(\bar{\eta}, \tau)\right]\delta(\eta_3) \end{aligned}$$

but has a net moment:

$$\begin{aligned} M_2 &= \int_V -\eta_1 f_3 dV \\ &= \int_V \eta_1 \frac{\partial}{\partial \eta_1}\left[\mu(\bar{\eta})\Delta u_1(\bar{\eta}, \tau)\right]\delta(\eta_3) d\eta_1 d\eta_2 d\eta_3 \\ &= -\int_\Sigma \mu \Delta u_1 d\Sigma = -\mu \overline{\Delta u_1} \Sigma \end{aligned}$$

Example: Slip on a fault

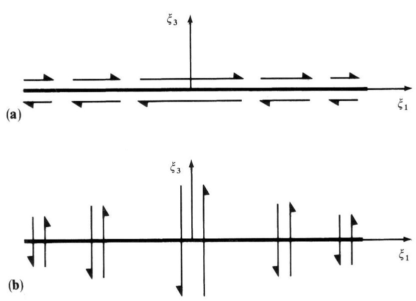
Interpretation of $f_3 = -\frac{\partial}{\partial \eta_1}\left[\mu(\bar{\eta})\Delta u_1(\bar{\eta},\tau)\right]\delta(\eta_3)$ distribution:

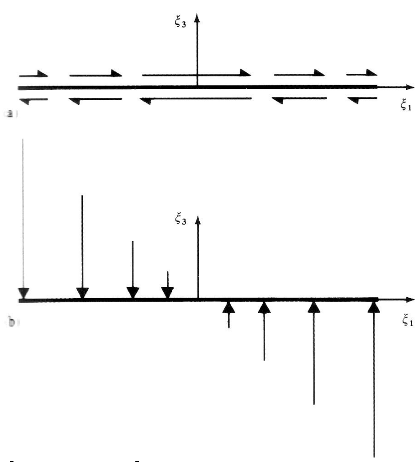


Fault slip is equivalent to a distribution of single couples (f_1) plus a distribution of single forces (f_3) that have the net effect of a opposing couple.

Example: Slip on a fault

Body-force equivalents are non-unique, different distributions produce the same radiation from slip on a fault.





For point source approximation → double couple.

Point source representation

Point source representation adequate if $\lambda \gg$ (length of Σ). Furthermore, if period $\gg$ source duration and origin at $t = 0$:

$$\begin{aligned} f_1(\bar{\eta}, \tau) &= -M_0 \delta(\eta_1)\delta(\eta_2)[\frac{\partial}{\partial \eta_3}\delta(\eta_3)]H(\tau) \\ f_2(\bar{\eta}, \tau) &= 0 \\ f_3(\bar{\eta}, \tau) &= -M_0[\frac{\partial}{\partial \eta_1}\delta(\eta_1)]\delta(\eta_2)\delta(\eta_3)H(\tau) \end{aligned}$$

where $H(\tau)$ is the Heaviside step function and

Seismic moment:

$$M_0 = \int_\Sigma \mu |\Delta \bar{u}| d\Sigma$$

Unit of M_0 is Nm.

Moment tensor representation

We have found

$$\begin{aligned} u_n(\bar{x}, t) &= \int_{-\infty}^{\infty} d\tau \int_V f_p^{\Delta \bar{u}}(\bar{\eta}, \tau) G_{np}(\bar{x}, t-\tau; \eta, 0) dV(\bar{\eta}) \\ &= \int_V f_p^{\Delta \bar{u}} * G_{np} dV \end{aligned}$$

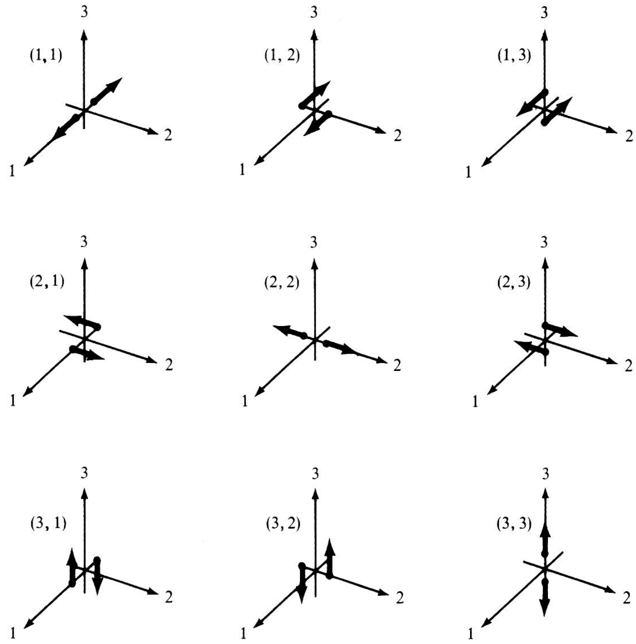
but also, earlier,

$$\begin{aligned} u_n(\bar{x}, t) &= \int_{-\infty}^{\infty} d\tau \int_\Sigma \Delta u_i(\bar{\xi}, \tau)\ \nu_j c_{ijpq} \frac{\partial}{\partial \xi_q} G_{np}(\bar{x}, t-\tau; \bar{\xi}, 0) d\Sigma(\bar{\xi}) \\ &= \int_\Sigma \Delta u_i \nu_j c_{ijpq} * \frac{\partial}{\partial \xi_q} G_{np} d\Sigma \end{aligned}$$

$\frac{\partial}{\partial \xi_q} G_{np}$ can be regarded as the response of a single couple with a force in the $\hat{\xi}_p$-direction and arm in the $\hat{\xi}_q$ direction.
$\Delta u_i \nu_j c_{ijpq}$ can be considered as the strength of the (p, q) couple.

Moment tensor representation

We therefore have 9 couples to obtain equivalent forces for a displacement discontinuity:



Moment tensor representation

Define the

Moment density tensor:

$$m_{pq} = \Delta u_i \nu_j c_{ijpq}$$

(Note that m_{pq} is symmetric.)

Then

$$u_n(\bar{x}, t) = \int_{\Sigma} m_{pq} * \frac{\partial}{\partial \xi_q} G_{np} d\Sigma$$

with in case of isotropy ($c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$) :

$$m_{pq} = \lambda \Delta u_i \nu_i \delta_{pq} + \mu(\Delta u_p \nu_q + \Delta u_q \nu_p)$$

Moment tensor representation

Examples:

(A) Fault slip with $\Delta\bar{u}$ parallel to plane Σ ($\rightarrow \Delta\bar{u}\cdot\hat{\nu} = 0$)

$$m_{pq} = \mu(\Delta u_p \nu_q + \Delta u_q \nu_p)$$

For our example with Σ plane with $\xi_3 = 0$, i.e. $\hat{\nu} = (0, 0, 1)$, and $\Delta\bar{u}$ in $\hat{\xi}_1$-direction $\Delta\bar{u} = (\Delta u_1, 0, 0)$:

$$m = \begin{pmatrix} 0 & 0 & \mu\Delta u_1 \\ 0 & 0 & 0 \\ \mu\Delta u_1 & 0 & 0 \end{pmatrix}$$

representing a double couple mechanism.

(B) Tension crack in plane with $\xi_3 = 0$ ($\rightarrow \hat{\nu} = (0, 0, 1)$), and $\Delta\bar{u}$ in $\hat{\xi}_3$-direction ($\rightarrow \Delta\bar{u} = (0, 0, \Delta u_3)$):

$$m = \begin{pmatrix} \lambda\Delta u_3 & 0 & 0 \\ 0 & \lambda\Delta u_3 & 0 \\ 0 & 0 & (\lambda+2\mu)\Delta u_3 \end{pmatrix}$$

Moment tensor representation

Define <u>moment tensor</u> M as:

$$M_{ij} = \int_{\Sigma} m_{ij}\, d\Sigma$$

so that

$$u_n = M_{pq} * G_{np,q}$$

where M is the representation for an effective point source.

Derivation of Green's function for a homogeneous, isotropic, unbounded medium

Statement of the problem

We try to find $\bar{u}(\bar{x}, t)$ for a point force in a homogeneous, unbounded, isotropic, elastic medium:

$$\rho \ddot{u}_i = f_i + (\lambda + \mu) u_{j,ji} + \mu u_{i,jj}$$

or

$$\rho \ddot{\bar{u}} = \bar{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \bar{u}) - \mu \nabla \times (\nabla \times \bar{u})$$

with a point force at $\bar{x} = \bar{0}$ in $\hat{x}_1$-direction:

$$\bar{f}(\bar{x}, t) = X_0(t)\delta(\bar{x})\hat{x}_1$$

So, solve the equation of motion for a point force:

$$\rho \ddot{\bar{u}} = X_0(t)\delta(\bar{x})\hat{x}_1 + (\lambda + 2\mu)\nabla(\nabla \cdot \bar{u}) - \mu \nabla \times (\nabla \times \bar{u})$$

The response of the medium to this force is:

$$u_n(\bar{x}, t) = X_0 * G_{n1}$$

Helmholtz decomposition

First, learn about Helmholtz decomposition.
Any vector field $\bar{f}(\bar{x})$ can be decomposed in terms of Helmholtz potentials Φ and $\bar{\Psi}$:

$$\bar{f} = \nabla\Phi + \nabla \times \bar{\Psi} \quad \text{with} \quad \nabla \cdot \bar{\Psi} = 0$$

Define

$$\nabla^2 \bar{F} = \bar{f}$$

using the definition of the vector Laplacian:

$$\nabla^2 \bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla \times (\nabla \times \bar{F})$$

$$\equiv \nabla\Phi + \nabla \times \bar{\Psi}$$

where

$$\Phi = \nabla \cdot \bar{F}$$

and

$$\bar{\Psi} = -\nabla \times \bar{F} \quad \textit{with} \quad \nabla \cdot \bar{\Psi} = -\nabla \cdot (\nabla \times \bar{F}) = 0$$

Helmholtz decomposition

Note:

$$\nabla \cdot \bar{f} = \nabla^2 \Phi$$

$$\begin{aligned} \nabla \times \bar{f} &= \nabla \times \nabla\Phi + \nabla \times \nabla \times \bar{\Psi} \\ &= \nabla(\nabla \cdot \bar{\Psi}) - \nabla^2 \bar{\Psi} = -\nabla^2 \bar{\Psi} \end{aligned}$$

With

$$\nabla^2 \bar{F} = \bar{f} = X_0(t)\delta(\bar{x})\hat{x}_1$$

we have

$$\begin{aligned} \nabla^2 F_1 &= X_0(t)\delta(\bar{x}) \\ \nabla^2 F_2 &= 0 \\ \nabla^2 F_3 &= 0 \end{aligned}$$

Finding the solution

$$\nabla^2 F_1 = X_0(t)\delta(\bar{x})$$

has form of Poisson's equation for fixed t.
The solution of

$$\nabla^2 g = -4\pi\delta(r)$$

is

$$g(r) = \frac{1}{r}$$

(e.g. Boas, section 13.8)
We therefore find

$$\begin{aligned} F_1 &= -X_0(t)\frac{1}{4\pi|\bar{x}|} \\ F_2 &= 0 \\ F_3 &= 0 \end{aligned}$$

Finding the solution

$\Phi = \nabla \cdot \bar{F}$:

$$\begin{aligned} \Phi(\bar{x},t) &= \nabla \cdot \left(-X_0(t)\frac{1}{4\pi|\bar{x}|}\hat{x}_1\right) \\ &= -\frac{X_0(t)}{4\pi}\frac{\partial}{\partial x_1}\frac{1}{|\bar{x}|} \end{aligned}$$

$\bar{\Psi} = -\nabla \times \bar{F}$:

$$\begin{aligned} \bar{\Psi}(\bar{x},t) &= \nabla \times \left(\frac{1}{4\pi|\bar{x}|}X_0(t)\hat{x}_1\right) \\ &= \frac{X_0(t)}{4\pi}\left(0, \frac{\partial}{\partial x_3}\frac{1}{|\bar{x}|}, -\frac{\partial}{\partial x_2}\frac{1}{|\bar{x}|}\right) \end{aligned}$$

Finding the solution, continued

After Helmholtz decomposition of $\bar{f}$ we can do a similar decomposition for $\bar{u}$:

$$\bar{u} = \nabla\phi + \nabla \times \bar{\psi} \quad \text{with} \quad \nabla \cdot \bar{\psi} = 0$$

Substitution in

$$\rho\ddot{\bar{u}} = \bar{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \bar{u}) - \mu\nabla \times (\nabla \times \bar{u})$$

gives (together with $\bar{f} = \nabla\Phi + \nabla \times \bar{\Psi}$):

$$\rho(\nabla\ddot{\phi} + \nabla\times\ddot{\bar{\psi}}) - \nabla\Phi - \nabla\times\bar{\Psi} - (\lambda+2\mu)\nabla(\nabla\cdot\nabla\phi) + \mu(\nabla\times\nabla\times\nabla\times\bar{\psi}) = \bar{0}$$

Finding the solution, continued

$$\rho(\nabla\ddot{\phi} + \nabla\times\ddot{\bar{\psi}}) - \nabla\Phi - \nabla\times\bar{\Psi} - (\lambda+2\mu)\nabla(\nabla\cdot\nabla\phi) + \mu(\nabla\times\nabla\times\nabla\times\bar{\psi}) = \bar{0}$$

By taking the divergence $(\nabla\cdot)$ we find

$$\rho\nabla^2\ddot{\phi} - \nabla^2\Phi - (\lambda + 2\mu)\nabla^2(\nabla^2\phi) = 0$$

or

$$\ddot{\phi} - \frac{(\lambda + 2\mu)}{\rho}\nabla^2\phi = \frac{\Phi}{\rho}$$

or

$$\ddot{\phi} - \alpha^2\nabla^2\phi = -\frac{X_0(t)}{4\pi\rho}\frac{\partial}{\partial x_1}\frac{1}{|\bar{x}|}$$

where $\nabla\phi$ is the P-wave component of $\bar{u}$ and $\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$, the P-wave speed.

Finding the solution, continued

$$\rho(\nabla\ddot{\phi}+\nabla\times\ddot{\bar{\psi}})-\nabla\Phi-\nabla\times\bar{\Psi}-(\lambda+2\mu)\nabla(\nabla\cdot\nabla\phi)+\mu(\nabla\times\nabla\times\nabla\times\bar{\psi})=\bar{0}$$

By taking the curl ($\nabla\times$) we find

$$\rho\nabla\times\nabla\times\ddot{\bar{\psi}}-\nabla\times\nabla\times\bar{\Psi}+\mu(\nabla\times\nabla\times\nabla\times\nabla\times\bar{\psi})=\bar{0}$$

or

$$\rho\ddot{\bar{\psi}}+\mu(\nabla\times\nabla\times\bar{\psi})=\bar{\Psi}$$

or

$$\ddot{\bar{\psi}}-\frac{\mu}{\rho}\nabla^2\bar{\psi}=\frac{\bar{\Psi}}{\rho}$$

or

$$\ddot{\bar{\psi}}-\beta^2\nabla^2\bar{\psi}=\frac{X_0(t)}{4\pi\rho}\left(0,\frac{\partial}{\partial x_3}\frac{1}{|\bar{x}|},-\frac{\partial}{\partial x_2}\frac{1}{|\bar{x}|}\right)$$

where $\nabla\times\bar{\psi}$ is the S-wave component of $\bar{u}$ and $\beta=\sqrt{\frac{\mu}{\rho}}$, the S-wave speed.

Solutions to scalar wave equations

Now we need to obtain the displacement $\bar{u}$ from the scalar potentials ϕ and $\bar{\psi}$. These potentials satisfy the wave equation. Find solutions to scalar wave equations:
(1) (see A & R, Box 4.1)

$$\ddot{g}-c^2\nabla^2 g=\delta(\bar{x})\delta(t)$$

$$g(\bar{x},t)=\frac{1}{4\pi c^2}\frac{\delta(t-|\bar{x}|/c)}{|\bar{x}|}$$

(2)

$$\ddot{g}-c^2\nabla^2 g=\delta(\bar{x}-\bar{\xi})\delta(t-\tau)$$

$$g(\bar{x},t)=\frac{1}{4\pi c^2}\frac{\delta\left(t-\tau-\frac{|\bar{x}-\bar{\xi}|}{c}\right)}{|\bar{x}-\bar{\xi}|}$$

Solutions to scalar wave equations

(3)

$$\ddot{g} - c^2\nabla^2 g = \delta(\bar{x} - \bar{\xi})f(t)$$

$$g(\bar{x}, t) = \frac{1}{4\pi c^2}\frac{f\left(t - \frac{|\bar{x}-\bar{\xi}|}{c}\right)}{|\bar{x} - \bar{\xi}|}$$

(4)

$$\ddot{g} - c^2\nabla^2 g = \Phi(\bar{x}, t)$$

with

$$\Phi(\bar{x}, t) = \int_{-\infty}^{\infty} d\tau \int_V \Phi(\bar{\xi}, \tau)\delta(\bar{x} - \bar{\xi})\delta(t - \tau)dV(\bar{\xi})$$

$$g(\bar{x}, t) = \frac{1}{4\pi c^2}\int_V \frac{\Phi(\bar{\xi}, t - \frac{|\bar{x}-\bar{\xi}|}{c})}{|\bar{x} - \bar{\xi}|}dV(\bar{\xi})$$

Solution for ϕ

For the scalar potential ϕ we had:

$$\ddot{\phi} - \alpha^2\nabla^2\phi = -\frac{X_0(t)}{4\pi\rho}\frac{\partial}{\partial x_1}\frac{1}{|\bar{x}|}$$

so we find the solution

$$\phi(\bar{x}, t) = -\frac{1}{(4\pi\alpha)^2\rho}\int_V \frac{X_0(t - \frac{|\bar{x}-\bar{\xi}|}{\alpha})}{|\bar{x} - \bar{\xi}|}\frac{\partial}{\partial \xi_1}\frac{1}{|\bar{\xi}|}dV(\bar{\xi})$$

With $|\bar{x} - \bar{\xi}| = \alpha\tau$ it can be shown (see A & R, Box 4.3) that

$$\phi(\bar{x}, t) = -\frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_1}\frac{1}{|\bar{x}|}\right)\int_0^{|\bar{x}|/\alpha} \tau X_0(t - \tau)d\tau$$

Solution for $\bar{\psi}$

For the vector potential $\bar{\psi}$ we had

$$\ddot{\bar{\psi}} - \beta^2 \nabla^2 \bar{\psi} = \frac{X_0(t)}{4\pi\rho} \left(0, \frac{\partial}{\partial x_3} \frac{1}{|\bar{x}|}, -\frac{\partial}{\partial x_2} \frac{1}{|\bar{x}|} \right)$$

which has the solution

$$\bar{\psi}(\bar{x}, t) = \frac{1}{4\pi\rho} \left(0, \frac{\partial}{\partial x_3} \frac{1}{|\bar{x}|}, -\frac{\partial}{\partial x_2} \frac{1}{|\bar{x}|} \right) \int_0^{|\bar{x}|/\beta} \tau X_0(t-\tau) d\tau$$

Solution for $\bar{u}$

With $\bar{u} = \nabla\phi + \nabla \times \bar{\psi}$, $|\bar{x}| = r$ (i.e. $\bar{\xi} = \bar{0}$), and some algebra, it follows:

$$\begin{aligned} u_i(\bar{x}, t) &= \frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau) d\tau + \frac{1}{4\pi\rho\alpha^2 r} \left(\frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1} \right) X_0(t \\ &+ \frac{1}{4\pi\rho\beta^2 r} \left(\delta_{i1} - \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1} \right) X_0(t - r/\beta) \end{aligned}$$

Now point force $X_0(t)$ in the $\hat{x}_j$ direction and using

$$\frac{x_i}{r} = \frac{\partial r}{\partial x_i} = \gamma_i \quad (direction \quad cosine)$$

$$\begin{aligned} u_i(\bar{x}, t) &= \frac{1}{4\pi\rho} (3\gamma_i\gamma_j - \delta_{ij}) \frac{1}{r^3} \int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau) d\tau + \frac{1}{4\pi\rho\alpha^2} \gamma_i\gamma_j \frac{1}{r} X_0(t - r/ \\ &+ \frac{1}{4\pi\rho\beta^2} (\delta_{ij} - \gamma_i\gamma_j) \frac{1}{r} X_0(t - r/\beta) \end{aligned}$$

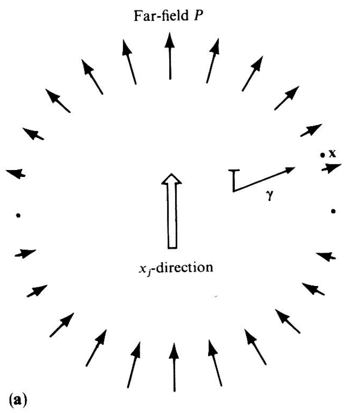
For short $X_0(t)$, 1st term $\propto \frac{1}{r^2}$ $\rightarrow$ near-field term (problem 4.1).
2nd & 3rd terms $\propto \frac{1}{r}$ $\rightarrow$ P-, S- far-field terms.

Far-field P-wave term due to point force

Far-field P-wave due to point force $X_0(t)$ in $\hat{x}_j$-direction:

$$u_i^P(\bar{x}, t) = \frac{1}{4\pi\rho\alpha^2}\gamma_i\gamma_j\frac{1}{r}X_0(t - r/\alpha)$$

- decays as $\frac{1}{r}$
- argument is $t - \frac{r}{\alpha}$: propagates with speed $\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$
- waveform proportional to $X_0(t)$ at retarded time
- u_i^P proportional to $\gamma_i \rightarrow$ longitudinal motion



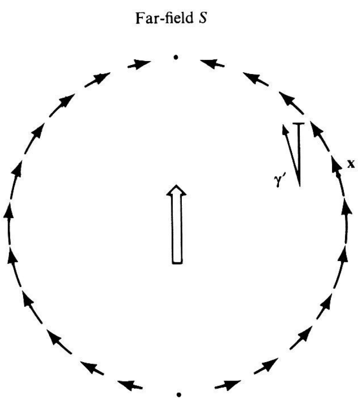
(a)

Far-field S-wave term due to point force

Far-field S-wave due to point force $X_0(t)$ in $\hat{x}_j$-direction:

$$u_i^S(\bar{x}, t) = \frac{1}{4\pi\rho\beta^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)\frac{1}{r}X_0(t - r/\beta)$$

- decays as $\frac{1}{r}$
- argument is $t - \frac{r}{\beta}$: propagates with speed $\beta = \sqrt{\frac{\mu}{\rho}}$
- waveform proportional to $X_0(t)$ at retarded time
- $\bar{u}^S$ perpendicular $\bar{\gamma} \rightarrow$ transverse motion



(b)

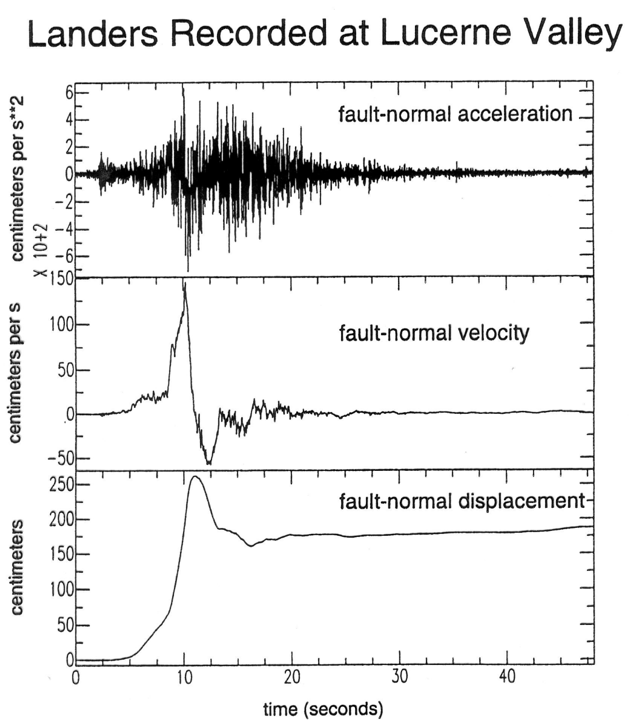
Near-field term due to point force

Near-field term due to point force $X_0(t)$ in $\hat{x}_j$-direction:

$$u_i^N(\bar{x}, t) = \frac{1}{4\pi\rho}(3\gamma_i\gamma_j - \delta_{ij})\frac{1}{r^3}\int_{r/\alpha}^{r/\beta}\tau X_0(t-\tau)d\tau$$

- Contributions from $\nabla\phi$ and $\nabla \times \bar{\psi}$
  $\rightarrow$ combination of P- and S-wave motion
- Waveform not proportional to $X_0(t)$

Near-field recording



M=7.2, 1992, Landers, California, earthquake at 3 km distance.

Green function for homogeneous, isotropic, elastic medium

$\bar{u}$ due to a point force $\bar{F}(t)\delta(\bar{x}-\bar{\xi})$ for a hom., iso., el. medium:

$$\begin{aligned} u_n(\bar{x},t) &= \frac{1}{4\pi\rho}\left(3\gamma_n\gamma_p-\delta_{np}\right)\frac{1}{r^3}\int_{r/\alpha}^{r/\beta}\tau F_p(t-\tau)d\tau \\ &+ \frac{1}{4\pi\rho\alpha^2}\gamma_n\gamma_p\frac{1}{r}F_p(t-r/\alpha) \\ &+ \frac{1}{4\pi\rho\beta^2}\left(\delta_{np}-\gamma_n\gamma_p\right)\frac{1}{r}F_p(t-r/\beta) \end{aligned}$$

where $r=|\bar{x}-\bar{\xi}|$ and $\gamma_i=\frac{x_i-\xi_i}{r}$, i.e. $\hat{\gamma}$ is unit vector from $\bar{\xi}$ to $\bar{x}$.
Since $u_n(\bar{x},t)=F_p * G_{np}$ this means that we have the Green function:

$$\begin{aligned} G_{np}(\bar{x},t;\bar{\xi},0) &= \frac{1}{4\pi\rho r^3}\left(3\gamma_n\gamma_p-\delta_{np}\right)\int_{r/\alpha}^{r/\beta}\tau\delta(t-\tau)d\tau \\ &+ \frac{1}{4\pi\rho\alpha^2}\gamma_n\gamma_p\frac{1}{r}\delta(t-r/\alpha) \\ &+ \frac{1}{4\pi\rho\beta^2}\left(\delta_{np}-\gamma_n\gamma_p\right)\frac{1}{r}\delta(t-r/\beta) \end{aligned}$$

We have the Green function in a homogeneous, isotropic, elastic medium.

Now find the displacement field $\bar{u}$ due to dislocation source.

Displacement field $\bar{u}$ due to dislocation source

For a displacement discontinuity we had

$$u_n(\bar{x}, t) = \int_{\Sigma} \Delta u_i \nu_j c_{ijpq} * \frac{\partial}{\partial \xi_q} G_{np} d\Sigma$$

or

$$u_n(\bar{x}, t) = \int_{\Sigma} m_{pq} * G_{np,q} d\Sigma$$

To obtain far-field expression due to dislocation source we need far-field expression of $G_{np,q}$.
The P-wave component of the Green function is:

$$G^P_{np} = \frac{1}{4\pi\rho\alpha^2} \gamma_n \gamma_p \frac{1}{r} \delta(t - r/\alpha)$$

Thus

$$\begin{aligned} G^P_{np,q} &= \frac{1}{4\pi\rho\alpha^2} \left[\gamma_{n,q}\gamma_p \frac{1}{r}\delta(t - r/\alpha) + \gamma_n \gamma_{p,q} \frac{1}{r} \delta(t - r/\alpha) \right. \\ &+ \left. \gamma_n \gamma_p \left(\frac{1}{r}\right)_{,q} \delta(t - r/\alpha) + \gamma_n \gamma_p \frac{1}{r} \left[\delta(t - r/\alpha)\right]_{,q} \right] \end{aligned}$$

Displacement field $\bar{u}$ due to dislocation source

$$\begin{aligned} G^P_{np,q} &= \frac{1}{4\pi\rho\alpha^2} \left[\gamma_{n,q}\gamma_p \frac{1}{r}\delta(t - r/\alpha) + \gamma_n \gamma_{p,q} \frac{1}{r} \delta(t - r/\alpha) \right. \\ &+ \left. \gamma_n \gamma_p \left(\frac{1}{r}\right)_{,q} \delta(t - r/\alpha) + \gamma_n \gamma_p \frac{1}{r} \left[\delta(t - r/\alpha)\right]_{,q} \right] \end{aligned}$$

With

$$\gamma_{i,q} = \frac{\partial \gamma_i}{\partial \xi_q} = \frac{\gamma_i \gamma_q - \delta_{iq}}{r}$$

and

$$\frac{\partial r}{\partial \xi_q} = -\gamma_q$$

1st term: $\propto \frac{1}{r^2}$: $(\gamma_{n,q} \frac{1}{r})$
2nd term: $\propto \frac{1}{r^2}$: $(\gamma_{p,q} \frac{1}{r})$
3rd term: $\propto \frac{1}{r^2}$: $(-\frac{1}{r^2} \cdot -\gamma_q)$
4th term: $\propto \frac{1}{r}$: $\frac{\partial}{\partial \xi_q} \delta(t - r/\alpha) = \dot{\delta}(t - r/\alpha) \frac{-1}{\alpha} \frac{\partial r}{\partial \xi_q} = \dot{\delta}(t - r/\alpha) \frac{\gamma_q}{\alpha}$

Displacement field $\bar{u}$ due to dislocation source

Neglect $O(\frac{1}{r^2})$ terms for far-field approximation:

$$G^P_{np,q} \simeq \frac{\gamma_n \gamma_p \gamma_q}{4\pi\rho\alpha^3} \frac{1}{r} \dot{\delta}(t - r/\alpha)$$

For a point dislocation source with $u_n(\bar{x}, t) = M_{pq} * G_{np,q}$
the far-field P-wave displacement field is:

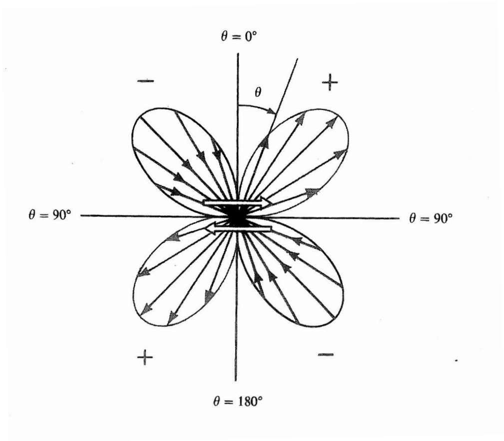
$$u^P_n(\bar{x}, t) = \frac{\gamma_n \gamma_p \gamma_q}{4\pi\rho\alpha^3 r} \dot{M}_{pq}(t - r/\alpha)$$

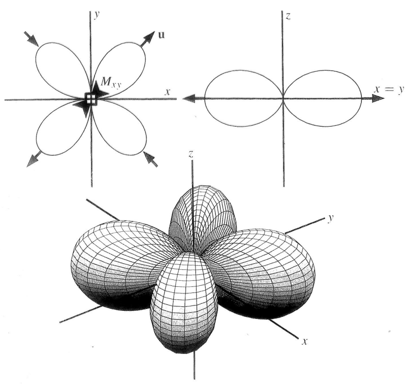
where $\dot{M}_{pq}(t - r/\alpha)$ is the time derivative of the moment tensor.

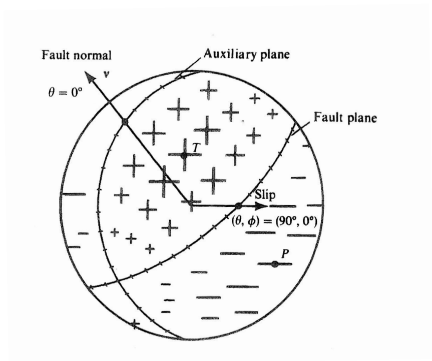
Analogously, the far-field S-wave displacement field is:

$$u^S_n(\bar{x}, t) = \frac{(\delta_{np} - \gamma_n \gamma_p)\gamma_q}{4\pi\rho\beta^3 r} \dot{M}_{pq}(t - r/\beta)$$

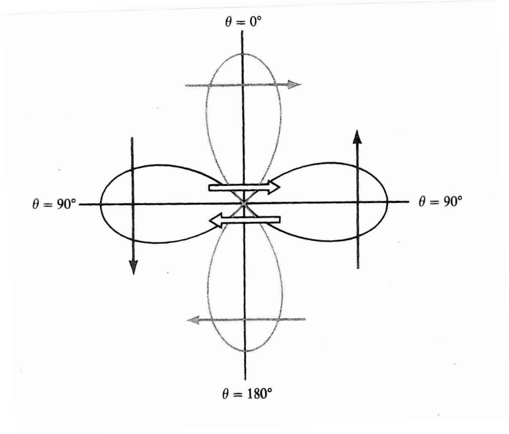
Double-couple P-wave radiation pattern

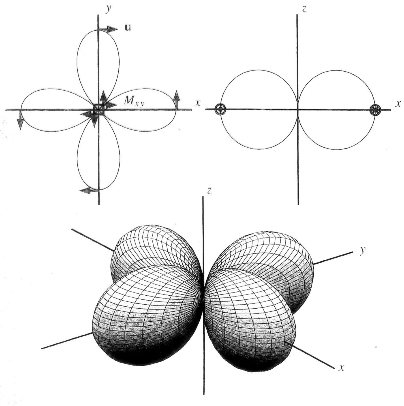


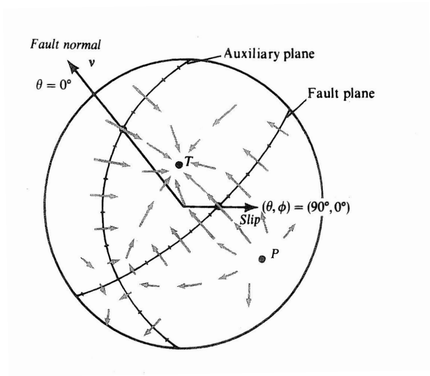




Double-couple S-wave radiation pattern







P- and S- wave displacement fields

We had

$$
\begin{aligned}
u_n^P(\bar{x}, t) &= \frac{\gamma_n \gamma_p \gamma_q}{4\pi\rho\alpha^3 r} \dot{M}_{pq}(t - r/\alpha) \\
u_n^S(\bar{x}, t) &= \frac{(\delta_{np} - \gamma_n \gamma_p)\gamma_q}{4\pi\rho\beta^3 r} \dot{M}_{pq}(t - r/\beta)
\end{aligned}
$$

With

$$M_{pq}(t) = \mu(\nu_p \Delta u_q(t) + \nu_q \Delta u_p(t))\Sigma$$

we find that the far-field displacement field $u_n(\bar{x}, t)$ is proportional to $\Delta\dot{u}_i$, the time derivative of the displacement at the source.

And a similar $\Delta\dot{u}_i$-dependence when the moment tensor density is used $(m_{pq}(t) = \int_\Sigma \Delta u_i(\bar{\xi}, t)\nu_j c_{ijkl} d\Sigma)$.

P- and S- wave displacement fields

Isolating time dependence: $M_{pq}(t) = M_{pq} f(t)$, we have

$$u_n^P(\bar{x}, t) = \frac{1}{4\pi\rho\alpha^3} \frac{1}{r} \gamma_n \gamma_p \gamma_q M_{pq} \dot{f}(t - r/\alpha)$$

and we can recognize the following factors:

$$\frac{1}{4\pi\rho\alpha^3} \frac{1}{r} \quad \textit{geometrical spreading}$$

$$\gamma_n \gamma_p \gamma_q M_{pq} \quad \textit{radiation pattern}$$

$$\dot{f}(t - r/\alpha) \quad \textit{time dependence}$$

For u_n^S:

$$u_n^S(\bar{x}, t) = \frac{1}{4\pi\rho\beta^3} \frac{1}{r} (\delta_{np} - \gamma_n \gamma_p) \gamma_q M_{pq} \dot{f}(t - r/\beta)$$

$$\frac{1}{4\pi\rho\beta^3} \frac{1}{r} \quad \textit{geometrical spreading}$$

$$(\delta_{np} - \gamma_n \gamma_p) \gamma_q M_{pq} \quad \textit{radiation pattern}$$

$$\dot{f}(t - r/\beta) \quad \textit{time dependence}$$

Ray theory:

a high frequency approximation of the solution of the wave equation.

Ray theory: outline

We need to find the solution of

$$\rho \ddot{u}_i = \tau_{ij,j} = [c_{ijkl} u_{k,l}]_{,j}$$

everywhere outside the source region.
For general heterogeneous media this equation cannot be reduced to the wave equation.
We often use ray theory for inhomogeneous media specified by rays which are the normals to wavefronts.
In many applications it is sufficient to know the travel time and the amplitudes along the rays.
Outline of theory is presented for scalar wave equation:

$$\nabla^2 \phi - \frac{1}{c^2(\bar{x})} \ddot{\phi} = 0$$

Ray theory: outline

For a homogeneous medium the solution of the wave equation is

$$\phi(\bar{x}, t) = \phi_0 \left(t - \frac{x_1}{c} \right)$$

for a plane wave propagating in $\hat{x}_1$-direction
or

$$\phi(\bar{x}, t) = \frac{1}{r} \phi_0 \left(t - \frac{r}{c} \right)$$

for a spherical wave.

(1) Waveform remains same ($\phi_0(t)$) and propagates with constant velocity c.
(2) The amplitude varies as geometrical spreading occurs.

We look for a similar solution for an inhomogeneous medium:
a pulse propagating with local velocity $c(\bar{x})$ without distortion.

Ray theory: solve wave equation in frequency domain

Using Fourier transform w.r.t. time, and its inverse:

$$\phi(\bar{x},\omega) = \int_{-\infty}^{\infty} \phi(\bar{x},t)e^{i\omega t}dt$$

$$\phi(\bar{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\bar{x},\omega)e^{-i\omega t}d\omega$$

we obtain in frequency domain:

$$\nabla^2\phi + \frac{\omega^2}{c^2}\phi = 0$$

Take as trial solution:

$$\phi(\bar{x},\omega) = \phi_0(\omega)A(\bar{x})e^{i\omega T(\bar{x})}$$

Important features: (1) separation of frequency and spatial dependence, and (2) linear frequency dependence of phase.

Ray theory: solve wave equation in frequency domain

Then

$$\nabla\phi = \phi_0\left(\nabla A e^{i\omega T} + i\omega A\nabla T e^{i\omega T}\right)$$

and

$$\begin{aligned}\nabla^2\phi &= \phi_0\Big(\nabla^2 A e^{i\omega T} + 2i\omega(\nabla A\cdot\nabla T)e^{i\omega T} \\ &+ i\omega A\nabla^2 T e^{i\omega T} - \omega^2 A|\nabla T|^2 e^{i\omega T}\Big)\end{aligned}$$

Substitution in wave equation and division by $\phi_0 e^{i\omega T}$:

$$\left[\nabla^2 A - \omega^2 A|\nabla T|^2 + \frac{\omega^2}{c^2}A\right] + i\left[2\omega(\nabla A\cdot\nabla T) + \omega A\nabla^2 T\right] = 0$$

Real part and imaginary part have to be zero.

Ray theory: eikonal equation

Real part:

$$\nabla^2 A - \omega^2 A |\nabla T|^2 + \frac{\omega^2}{c^2} A = 0$$

For high frequency: 2nd and 3rd terms are dominant
$\longrightarrow$ high frequency approximation yields

Eikonal equation:

$$|\nabla T(\bar{x})|^2 = \frac{1}{c^2(\bar{x})}$$

giving the travel time $T(x)$.

Ray theory: transport equation

Imaginary part:

$$2\omega(\nabla A \cdot \nabla T) + \omega A \nabla^2 T = 0$$

This is the transport equation which allows the calculation of the amplitude $A(\bar{x})$ from travel time $T(\bar{x})$.

Ray theory: interpretation of the solution

We had

$$\phi(\bar{x},\omega) = \phi_0(\omega)A(\bar{x})e^{i\omega T(\bar{x})}$$

where $\phi_0(\omega)$ is the source spectrum:

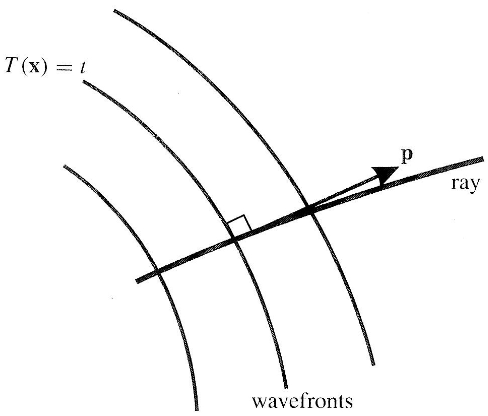
$$\phi_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_0(\omega)e^{-i\omega t}d\omega$$

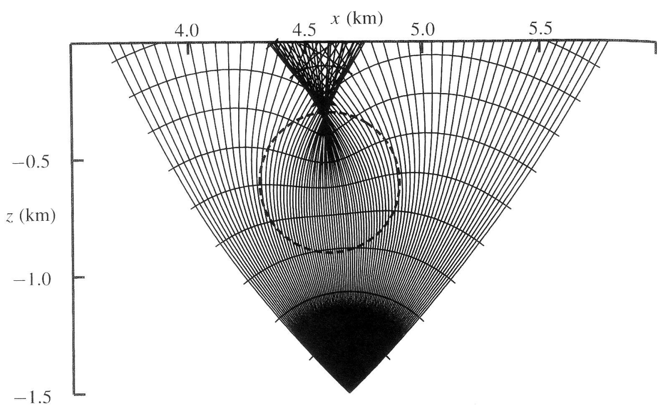
We obtain for ϕ in the time domain:

$$\begin{aligned}\phi(\bar{x},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_0(\omega)A(\bar{x})e^{i\omega T(\bar{x})}e^{-i\omega t}d\omega \\ &= \frac{A(\bar{x})}{2\pi}\int_{-\infty}^{\infty}\phi_0(\omega)e^{-i\omega(t-T(\bar{x}))}d\omega \\ &= A(\bar{x})\phi_0\left(t-T(\bar{x})\right)\end{aligned}$$

In the high-frequency ray-geometrical limit the waveform does not change and the function $T(\bar{x})$ gives the travel time.
The lines $T(\bar{x}) = \textit{constant}$ are wavefronts.
Rays are defined as curves perpendicular to the wavefronts.

Rays and wavefronts





Kinematic ray tracing

Consider the implicit equation of a ray $\bar{x}(s)$ where s is the distance along the ray. The tangent is defined by

$$\hat{n} = \frac{d\bar{x}}{ds}$$

The tangent $\hat{n}$ is parallel to ∇T, with ∇T perpendicular to the wavefronts of $T(\bar{x}) = constant$.

With the eikonal equation $|\nabla T|^2 = \frac{1}{c^2}$
we obtain:

$$\frac{d\bar{x}}{ds} = c\nabla T$$

We define the slowness vector $\bar{p}$ as

$$\bar{p} = \nabla T = \frac{1}{c}\frac{d\bar{x}}{ds}$$

and

$$p = |\bar{p}| = \frac{1}{c}$$

Kinematic ray tracing

How do we obtain the ray path knowing $c(\bar{x})$?
We need to eliminate T in $\frac{d\bar{x}}{ds} = c\nabla T$ to obtain $\bar{x}(s)$ depending only on $c(\bar{x})$. This can be achieved by evaluating $\frac{d\nabla T}{ds}$:

$$\frac{d\nabla T}{ds} = \frac{d}{ds}\left(\frac{1}{c}\frac{d\bar{x}}{ds}\right)$$

The derivative to s is the projection of the gradient on tangent $\hat{n}$:

$$\frac{d}{ds} = \hat{n}\cdot\nabla = c\nabla T\cdot\nabla$$

$$\begin{aligned}\frac{d\nabla T}{ds} &= [c\nabla T\cdot\nabla](\nabla T) = \frac{c}{2}\nabla\left[(\nabla T)^2\right] = \frac{c}{2}\nabla\left(\frac{1}{c}\right)^2 \\ &= \frac{c}{2}\left(\frac{-2}{c^3}\nabla c\right) = -\frac{1}{c^2}\nabla c = \nabla\left(\frac{1}{c}\right)\end{aligned}$$

Kinematic ray tracing

Thus

$$\frac{d}{ds}\left(\frac{1}{c}\frac{d\bar{x}}{ds}\right) = \nabla\left(\frac{1}{c}\right)$$

or

$$\frac{d\bar{p}}{ds} = \nabla\left(\frac{1}{c}\right)$$

This equation can be solved by integration if the initial conditions $\bar{x}$ and $\hat{n} = \frac{d\bar{x}}{ds}$ are given.

Example: For a homogeneous medium ($c = const.$) we find

$$\frac{d^2\bar{x}}{ds^2} = \bar{0} \qquad \textit{therefore} \qquad \bar{x} = \bar{a}s + \bar{b}$$

where $\bar{a}$ and $\bar{b}$ are vector constants.
This means a straight line in the $\bar{a}$ direction with initial point $\bar{b}$.

Kinematic ray tracing

Now we want to find the travel time along the ray.
We already defined

$$\bar{p} = \nabla T = \frac{1}{c}\frac{d\bar{x}}{ds} = p\hat{n}$$

and

$$\frac{dT}{ds} = \hat{n}\cdot\nabla T = \hat{n}\cdot p\hat{n} = p(\hat{n}\cdot\hat{n}) = p = \frac{1}{c}$$

Or with

$$T = \int \frac{dT}{ds} ds$$

we have

$$T(\bar{p}) = \int_{\bar{x}(s)} \frac{1}{c(\bar{x})} ds$$

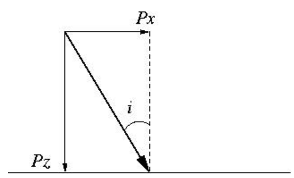
Kinematic ray tracing through 1-D model

Problem:
Find distance and travel time for a flat layered Earth model $c(z)$:

$$\frac{d\bar{p}}{ds} = \nabla\left(\frac{1}{c}\right) \rightarrow \begin{pmatrix} \frac{dp_x}{ds} \\ \frac{dp_y}{ds} \\ \frac{dp_z}{ds} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{1}{c} \\ \frac{\partial}{\partial y}\frac{1}{c} \\ \frac{\partial}{\partial z}\frac{1}{c} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial}{\partial z}\frac{1}{c} \end{pmatrix}$$

We find that $p_x = const$ and $p_y = const$ are along the ray.
With $\bar{p} = \frac{1}{c}\frac{d\bar{x}}{ds}$ and propagation in $x - z$ plane we have

$$\begin{aligned} p_x &= \frac{1}{c}\sin i \\ p_y &= 0 \\ p_z &= \frac{1}{c}\cos i \end{aligned}$$



Kinematic ray tracing through 1-D model

For the travel time T we have:

$$T = \int_{\bar{x}(s)} \frac{1}{c(\bar{x})} ds$$

so that we find

$$T(p_x) = 2\int_0^{z_{max}} \frac{1}{c(z)\sqrt{1 - c^2 p_x^2}} dz$$

And for the horizontal distance X we have:

$$X = \int_{\bar{x}(s)} dx$$

so that we find

$$X(p_x) = 2\int_0^{z_{max}} \frac{c p_x}{\sqrt{1 - c^2 p_x^2}} dz$$

Ray amplitude

Amplitude A is obtained from the transport equation:

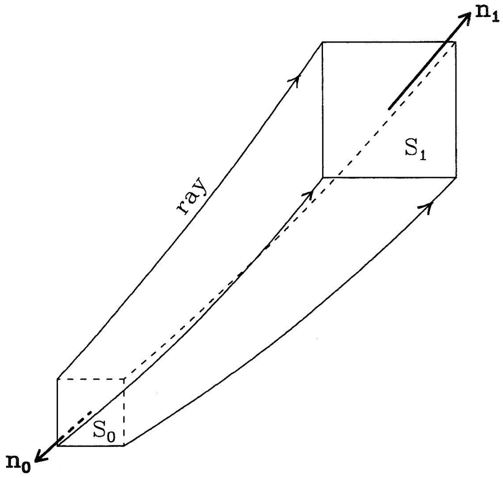
$$2(\nabla A \cdot \nabla T) + A\nabla^2 T = 0$$

multiply by A:

$$2A(\nabla A \cdot \nabla T) + A^2\nabla^2 T = 0$$

or

$$\nabla \cdot (A^2 \nabla T) = 0$$



Ray amplitude

Integrate over volume of tube of rays with end surfaces S_0 and S_1 at travel times t_0 and t_1, and apply Gauss's theorem:

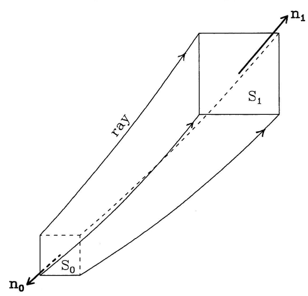
$$\int_S A^2 \nabla T \cdot \hat{n} dS = 0$$

S is the surface enclosing the volume, and $\hat{n}$ the outer unit vector normal to S.
The outer normal corresponding to S_0 is $-c\nabla T$.
The outer normal corresponding to S_1 is $c\nabla T$.
The outer normal along the tube is normal to the rays. For this part of S we have $\nabla T \cdot \hat{n} = 0$.

Thus:

$$\int_{S_0} cA^2 \nabla T \cdot \nabla T dS = \int_{S_1} cA^2 \nabla T \cdot \nabla T dS$$

Using the eikonal equation $|\nabla T|^2 = \frac{1}{c^2}$:



Ray amplitude

$$\int_{S_0} \frac{1}{c} A^2 dS = \int_{S_1} \frac{1}{c} A^2 dS$$

For a narrow tube we can approximate:

$$\left(\frac{1}{c} A^2\right)_0 \delta S_0 = \left(\frac{1}{c} A^2\right)_1 \delta S_1$$

or

$$\frac{1}{c} A^2 \delta S = constant$$

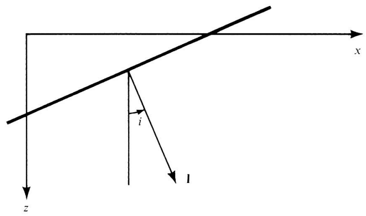
$\frac{\delta S_1}{\delta S_0}$ specifies the geometrical spreading.
$\longrightarrow$ The amplitude is inversely proportional to square root of the geometrical spreading.

Wavefield decomposition into plane waves

Plane waves

Plane wave propagates in direction $\hat{l}$ with speed c if

- at a fixed time the physical quantity is unchanged over each plane normal to $\hat{l}$,
- the plane propagates with speed c in direction $\hat{l}$.



Space-time dependence is given by:

$$t - \frac{\hat{l} \cdot \bar{x}}{c}$$

Slowness vector $\bar{s}$ is defined as:

$$\bar{s} \equiv \frac{\hat{l}}{c}$$

Plane waves

In Cartesian coordinate system:

$$\bar{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = \frac{1}{c} \begin{pmatrix} \sin i \\ 0 \\ \cos i \end{pmatrix}$$

for propagation in $x - z$ plane
Take

$$\bar{u}(\bar{x}, t) = \bar{u}(t - \bar{s} \cdot \bar{x})$$

to describe P- or S-wave displacement in isotropic medium. Note:

$$\frac{\partial \bar{u}}{\partial x_i} = -s_i \frac{\partial \bar{u}}{\partial t}$$

so

$$\nabla \cdot \bar{u} = -\bar{s} \cdot \dot{\bar{u}}$$

and

$$\nabla(\nabla \cdot \bar{u}) = \bar{s}(\bar{s} \cdot \ddot{\bar{u}})$$

Plane wave in wave equation

Substitution in wave equation for homogeneous, isotropic medium:

$$\rho\ddot{\bar{u}} = (\lambda + \mu)\nabla(\nabla \cdot \bar{u}) + \mu\nabla^2\bar{u}$$

$$\rho\ddot{\bar{u}} = (\lambda + \mu)(\bar{s} \cdot \ddot{\bar{u}})\bar{s} + \mu(\bar{s} \cdot \bar{s})\ddot{\bar{u}}$$

$$\rho\ddot{\bar{u}} - (\lambda + \mu)(\bar{s} \cdot \ddot{\bar{u}})\bar{s} - \mu(\bar{s} \cdot \bar{s})\ddot{\bar{u}} = \bar{0}$$

(a) Take scalar (dot) product with $\bar{s}$ (component of $\bar{u}$ parallel to $\bar{s}$):

$$\rho(\ddot{\bar{u}} \cdot \bar{s}) - (\lambda + \mu)(\bar{s} \cdot \ddot{\bar{u}})(\bar{s} \cdot \bar{s}) - \mu(\bar{s} \cdot \bar{s})(\ddot{\bar{u}} \cdot \bar{s}) = 0$$

$$\rho(\ddot{\bar{u}} \cdot \bar{s}) - \frac{(\lambda + \mu)}{c^2}(\bar{s} \cdot \ddot{\bar{u}}) - \frac{\mu}{c^2}(\ddot{\bar{u}} \cdot \bar{s}) = 0$$

$$\left(\rho - \frac{\lambda + 2\mu}{c^2}\right)(\ddot{\bar{u}} \cdot \bar{s}) = 0$$

So with $\ddot{\bar{u}} \cdot \bar{s} \neq 0$ for $\bar{u}$ parallel to propagation direction: $c^2 = \frac{\lambda+2\mu}{\rho}$.

$$\rho\ddot{\bar{u}} - (\lambda + \mu)(\bar{s} \cdot \ddot{\bar{u}})\bar{s} - \mu(\bar{s} \cdot \bar{s})\ddot{\bar{u}} = \bar{0}$$

(b) Take vector (cross) product with $\bar{s}$ (component of $\bar{u}$ perpendicular to $\bar{s}$):

$$\rho(\ddot{\bar{u}} \times \bar{s}) - (\lambda + \mu)(\bar{s} \cdot \ddot{\bar{u}})(\bar{s} \times \bar{s}) - \mu(\bar{s} \cdot \bar{s})(\ddot{\bar{u}} \times \bar{s}) = \bar{0}$$

$$\left(\rho - \frac{\mu}{c^2}\right)(\ddot{\bar{u}} \times \bar{s}) = \bar{0}$$

So with $\ddot{\bar{u}} \times \bar{s} \neq \bar{0}$ for $\bar{u}$ perpendicular to propagation direction: $c^2 = \frac{\mu}{\rho}$.

Separation of variables ↔ plane wave decomposition

Separation of variables of wave equation is similar to plane wave decomposition of wave field.

$$\ddot{\phi} = \alpha^2 \nabla^2 \phi$$

$$\phi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

$$\frac{\partial \phi}{\partial x} = \frac{dX}{dx} YZT$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{d^2 X}{dx^2} YZT = \frac{1}{X}\frac{d^2 X}{dx^2}\phi$$

With similar expressions for other derivatives:

$$\frac{1}{T}\frac{d^2 T}{dt^2} = \alpha^2 \left(\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} \right)$$

Separation of variables ↔ plane wave decomposition

$$\frac{1}{T}\frac{d^2 T}{dt^2} = \alpha^2 \left(\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} \right)$$

Differentiate w.r.t. t to see that:

$$\frac{1}{T}\frac{d^2 T}{dt^2} = constant = -\omega^2$$

$$\frac{d^2 T}{dt^2} + \omega^2 T = 0$$

$$T \propto e^{\pm i\omega t}$$

Similarly

$$\frac{1}{X}\frac{d^2 X}{dx^2} = constant = -k_x^2$$

$$\frac{d^2 X}{dx^2} + k_x^2 X = 0 \quad and \quad X \propto e^{\pm i k_x x}$$

and

$$\frac{d^2 Y}{dy^2} + k_y^2 Y = 0 \quad and \quad Y \propto e^{\pm i k_y y}$$

Separation of variables $\leftrightarrow$ plane wave decomposition

$$\frac{d^2 Z}{dz^2} + k_z^2 Z = 0 \quad and \quad Z \propto e^{\pm i k_z z}$$

where

$$k_z^2 = \frac{\omega^2}{\alpha^2} - k_x^2 - k_y^2$$

Solutions are of type

$$e^{i(\bar{k}\cdot\bar{x} - \omega t)}$$

with $\bar{k} = (k_x, k_y, k_z)$ and $|\bar{k}| = \omega/\alpha$.
and $\omega =$ (angular) frequency
and $\bar{k} =$ wavenumber vector.
Alternatively:

$$e^{i\omega(\bar{s}\cdot\bar{x} - t)}$$

with $\bar{s} =$ slowness vector.

Separation of variables $\leftrightarrow$ plane wave decomposition

General solutions:

$$\phi(x, y, z, t) = \iiint_{-\infty}^{\infty} \Phi(k_x, k_y, \omega) e^{i(k_x x + k_y y + \sqrt{\omega^2/\alpha^2 - k_x^2 - k_y^2} z - \omega t)} dk_x dk_y d\omega$$

- Solution given in terms of horizontal wavenumbers k_x and k_y and frequency.
- Amplitude $\Phi(k_x, k_y, \omega)$ depends on source excitation.
- Evaluations of triple integral often with approximations or done numerically

Plane wave reflection and transmission coefficients

Plane waves and elastic potentials

P- and S-components of the wave field are separated through their elastic potentials.

P-wave: $\bar{u}^P = \nabla\phi$ with $\ddot{\phi} = \alpha^2\nabla^2\phi$
For propagation in $x - z$ plane $\phi(x, z, t)$:

$$\bar{u}^P = \begin{pmatrix} \partial\phi/\partial x \\ 0 \\ \partial\phi/\partial z \end{pmatrix}$$

S-wave: $\bar{u}^S = \nabla \times \bar{\psi}$ with $\nabla \cdot \bar{\psi} = 0$ and $\ddot{\bar{\psi}} = \beta^2\nabla^2\bar{\psi}$
For propagation in $x - z$ plane $\bar{\psi}(x, z, t)$:

$$\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_z}{\partial z} = 0$$

Plane waves and elastic potentials: SH- and SV-component

If y-component of $\bar{u}^S$ is zero: SV-wave

$$u_y^{SV} = \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x} = 0$$

$\bar{u}^{SV}$ can be expressed by scalar potential ψ: $\bar{\psi} = (0, \psi, 0)$

$$\bar{u}^{SV} = \begin{pmatrix} -\partial\psi/\partial z \\ 0 \\ \partial\psi/\partial x \end{pmatrix}$$

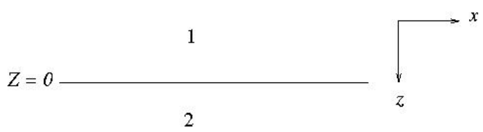
with $\ddot{\psi} = \beta^2 \nabla^2 \psi$
If x- and z-components of $\bar{u}^S$ are zero: SH-wave

$$\bar{u}^{SH} = \begin{pmatrix} 0 \\ u^{SH} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$$

with $\ddot{u}^{SH} = \beta^2 \nabla^2 u^{SH}$

Reflection and transmission: boundary conditions

We still need to describe what happens at a sharp transition, a boundary between two media.
Boundary conditions for displacement and traction:



Solid/solid in welded contact

- u_x, u_y, u_z continuous at discontinuity
- $\bar{T}(\hat{z})$ continuous $\longrightarrow \tau_{zx}, \tau_{zy}, \tau_{zz}$ continuous

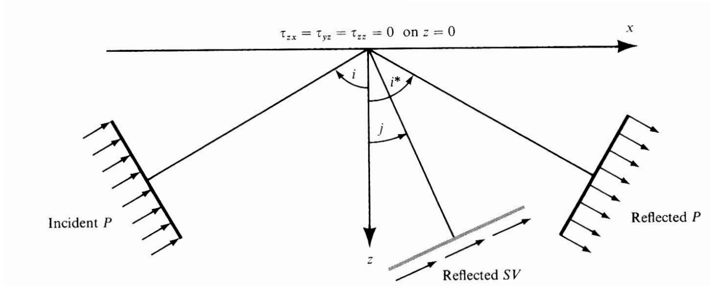
Solid/fluid

- u_z continuous at discontinuity
- $\tau_{zx}, \tau_{zy}, \tau_{zz}$ continuous

Solid/vacuum

- $\tau_{zx} = \tau_{zy} = \tau_{zz} = 0$

P-SV Reflection at free surface



For P-wave:

$$\bar{u}^P = \left(\frac{\partial \phi}{\partial x}, 0, \frac{\partial \phi}{\partial z}\right)$$

Use $\tau_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} = \lambda\theta\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$\begin{aligned}
\tau^P_{zx} &= 2\mu e_{zx} = 2\mu\left[\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\right] = 2\mu\frac{\partial^2\phi}{\partial x \partial z} \\
\tau^P_{zy} &= 2\mu e_{zy} = 0 \\
\tau^P_{zz} &= \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{zz} = \lambda\nabla^2\phi + 2\mu\frac{\partial^2\phi}{\partial z^2}
\end{aligned}$$

P-SV Reflection at free surface

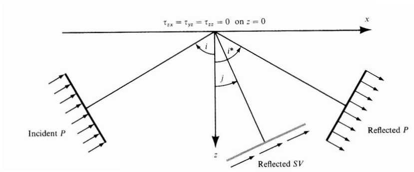
For SV-wave:

$$\begin{aligned}
\bar{u}^{SV} &= \left(\frac{-\partial\psi}{\partial z}, 0, \frac{\partial\psi}{\partial x}\right) \\
\tau^{SV}_{zx} &= \mu\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right) \\
\tau^{SV}_{zy} &= 0 \\
\tau^{SV}_{zz} &= \lambda\left(\frac{-\partial^2\psi}{\partial x \partial z} + 0 + \frac{\partial^2\psi}{\partial z \partial x}\right) + 2\mu\frac{\partial^2\psi}{\partial z \partial x} = 2\mu\frac{\partial^2\psi}{\partial z \partial x}
\end{aligned}$$

For SH-wave:

$$\begin{aligned}
\bar{u}^{SH} &= (0, v, 0) \\
\tau^{SH}_{zx} &= 0 \\
\tau^{SH}_{zy} &= \mu\frac{\partial v}{\partial z} \\
\tau^{SH}_{zz} &= 0
\end{aligned}$$

P-SV Reflection at free surface



Slowness vector of incident P-wave:

$$\bar{s}^{P,inc} = \left(\frac{\sin i}{\alpha}, 0, -\frac{\cos i}{\alpha} \right)$$

Slowness vector of reflected P-wave:

$$\bar{s}^{P,refl} = \left(\frac{\sin i^*}{\alpha}, 0, \frac{\cos i^*}{\alpha} \right)$$

Slowness vector of reflected SV-wave:

$$\bar{s}^{SV,refl} = \left(\frac{\sin j}{\beta}, 0, \frac{\cos j}{\beta} \right)$$

$\tau_{yz} = 0$ of incident P-wave $\rightarrow$ no SH-waves

P-SV Reflection at free surface

Total P-wave field from $\phi = \phi^{inc} + \phi^{refl}$

$$\phi^{inc} = Ae^{i\omega\left(\frac{\sin i}{\alpha}x - \frac{\cos i}{\alpha}z - t\right)}$$

$$\phi^{refl} = Be^{i\omega\left(\frac{\sin i^*}{\alpha}x + \frac{\cos i^*}{\alpha}z - t\right)}$$

SV-wave field from $\psi = \psi^{refl}$

$$\psi^{refl} = Ce^{i\omega\left(\frac{\sin j}{\beta}x + \frac{\cos j}{\beta}z - t\right)}$$

A, B, and C are constants per wave.

P-SV Reflection at free surface

Displacement $\bar{u}$ not constrained on $z = 0$.
Boundary condition for $z = 0$: $\tau_{zx} = \tau_{zz} = 0$.
For $z = 0$: τ_{zx} and τ_{zz} determined by three contributions with factors:

$$e^{i\omega\left(\frac{\sin i}{\alpha}x - t\right)}, \; e^{i\omega\left(\frac{\sin i^*}{\alpha}x - t\right)}, \text{and } e^{i\omega\left(\frac{\sin j}{\beta}x - t\right)}$$

Arguments must be the same for all x and all t $\longrightarrow$

$$i = i^* \quad and \quad \frac{\sin i}{\alpha} = \frac{\sin j}{\beta} \quad (Snell's\ law)$$

horizontal slowness (s_x), or ray parameter (p) is constant:

$$s_x = p = \frac{\sin i}{\alpha} = \frac{\sin j}{\beta}$$

P-SV Reflection at free surface

For P-wave:

$$\bar{u}^P = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ 0 \\ \frac{\partial\phi}{\partial z} \end{pmatrix} = \begin{pmatrix} i\omega p\phi \\ 0 \\ \frac{\partial\phi}{\partial z} \end{pmatrix}$$

$$\bar{T}^P(\hat{x}_z) = \begin{pmatrix} 2\mu\frac{\partial^2\phi}{\partial z\partial x} \\ 0 \\ \lambda\nabla^2\phi + 2\mu\frac{\partial^2\phi}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2\rho\beta^2 i\omega p\frac{\partial\phi}{\partial z} \\ 0 \\ -\rho(1 - 2\beta^2p^2)\omega^2\phi \end{pmatrix}$$

For SV-wave:

$$\bar{u}^{SV} = \begin{pmatrix} -\frac{\partial\psi}{\partial z} \\ 0 \\ \frac{\partial\psi}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{\partial\psi}{\partial z} \\ 0 \\ i\omega p\psi \end{pmatrix}$$

$$\bar{T}^{SV}(\hat{x}_z) = \begin{pmatrix} \mu\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right) \\ 0 \\ 2\mu\frac{\partial^2\psi}{\partial z\partial x} \end{pmatrix} = \begin{pmatrix} \rho(1 - 2\beta^2p^2)\omega^2\psi \\ 0 \\ 2\rho\beta^2 i\omega p\frac{\partial\psi}{\partial z} \end{pmatrix}$$

P-SV Reflection at free surface

On $z = 0$:

$$\tau_{zx} = 2\rho\beta^2 i\omega p \left(\frac{\partial \phi^{inc}}{\partial z} + \frac{\partial \phi^{refl}}{\partial z} \right) + \rho(1 - 2\beta^2 p^2)\omega^2 \psi^{refl} = 0$$

$$\tau_{zz} = -\rho(1 - 2\beta^2 p^2)\omega^2 \left(\phi^{inc} + \phi^{refl} \right) + 2\rho\beta^2 i\omega p \frac{\partial \psi^{refl}}{\partial z} = 0$$

or

$$2\rho\beta^2 p \frac{\cos i}{\alpha}(A - B) + \rho(1 - 2\beta^2 p^2) C = 0$$

$$-\rho(1 - 2\beta^2 p^2)(A + B) - 2\rho\beta^2 p \frac{\cos j}{\beta} C = 0$$

giving

$$\frac{B}{A} = \frac{4\beta^4 \ p^2 \ \frac{\cos i}{\alpha} \ \frac{\cos j}{\beta} - (1 - 2\beta^2 p^2)^2}{4\beta^4 \ p^2 \ \frac{\cos i}{\alpha} \ \frac{\cos j}{\beta} + (1 - 2\beta^2 p^2)^2}$$

and

$$\frac{C}{A} = \frac{-4\beta^2 \ p \ \frac{\cos i}{\alpha}(1 - 2\beta^2 p^2)}{4\beta^4 \ p^2 \ \frac{\cos i}{\alpha} \ \frac{\cos j}{\beta} + (1 - 2\beta^2 p^2)^2}$$

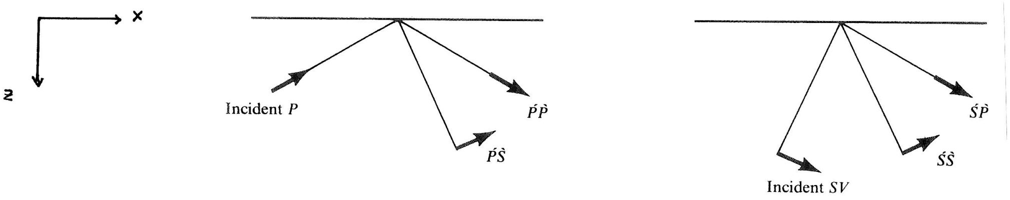
P-SV Reflection at free surface

Note that $\frac{B}{A}$ and $\frac{C}{A}$ are 'reflection coefficients' of the potentials.
Displacement amplitudes are derived from potentials:
P-wave displacement: amplification factor $= \left| \frac{\omega \phi}{\alpha} \right|$
S-wave displacement: amplification factor $= \left| \frac{\omega \psi}{\beta} \right|$
With signs as chosen in figure and table, we find:

$$\acute{P}\grave{P} = \frac{-\left(\frac{1}{\beta^2} - 2p^2 \right)^2 + 4p^2 \ \frac{\cos i}{\alpha} \ \frac{\cos j}{\beta}}{\left(\frac{1}{\beta^2} - 2p^2 \right)^2 + 4p^2 \ \frac{\cos i}{\alpha} \ \frac{\cos j}{\beta}}$$

$$\acute{P}\grave{S} = \frac{4\frac{\alpha}{\beta} \ p \ \frac{\cos i}{\alpha} \left(\frac{1}{\beta^2} - 2p^2 \right)}{\left(\frac{1}{\beta^2} - 2p^2 \right)^2 + 4p^2 \ \frac{\cos i}{\alpha} \ \frac{\cos j}{\beta}}$$

P-SV Reflection at free surface



Incident wave Type	Incident wave Displacement	Scattered waves Type	Scattered waves Displacement
Upgoing P	$S(\sin i, 0, -\cos i)\exp\left[i\omega\left(\frac{\sin i}{\alpha}x - \frac{\cos i}{\alpha}z - t\right)\right]$	Downgoing P	$S(\sin i, 0, \cos i)\acute{P}\grave{P}\exp\left[i\omega\left(\frac{\sin i}{\alpha}x + \frac{\cos i}{\alpha}z - t\right)\right]$
		Downgoing SV	$S(\cos j, 0, -\sin j)\acute{P}\grave{S}\exp\left[i\omega\left(\frac{\sin j}{\beta}x + \frac{\cos j}{\beta}z - t\right)\right]$
Upgoing SV	$S(\cos j, 0, \sin j)\exp\left[i\omega\left(\frac{\sin j}{\beta}x - \frac{\cos j}{\beta}z - t\right)\right]$	Downgoing P	$S(\sin i, 0, \cos i)\acute{S}\grave{P}\exp\left[i\omega\left(\frac{\sin i}{\alpha}x + \frac{\cos i}{\alpha}z - t\right)\right]$
		Downgoing SV	$S(\cos j, 0, -\sin j)\acute{S}\grave{S}\exp\left[i\omega\left(\frac{\sin j}{\beta}x + \frac{\cos j}{\beta}z - t\right)\right]$

P-SV Reflection at free surface

For an SV-wave incident on the free surface:

$$\acute{S}\grave{P} = \frac{4\frac{\beta}{\alpha}\, p\, \frac{\cos j}{\beta}\left(\frac{1}{\beta^2} - 2p^2\right)}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\, \frac{\cos i}{\alpha}\, \frac{\cos j}{\beta}}$$

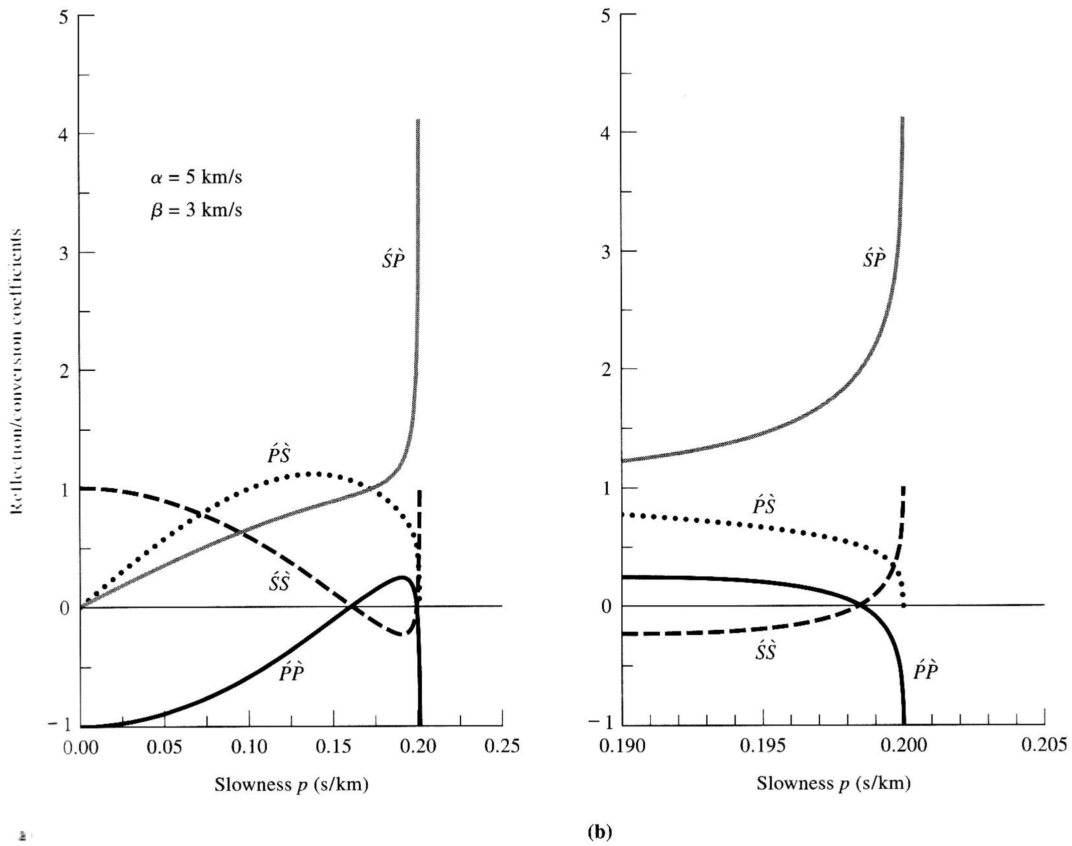
$$\acute{S}\grave{S} = \frac{\left(\frac{1}{\beta^2} - 2p^2\right)^2 - 4p^2\, \frac{\cos i}{\alpha}\, \frac{\cos j}{\beta}}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\, \frac{\cos i}{\alpha}\, \frac{\cos j}{\beta}}$$

We call

$$\begin{pmatrix} \acute{P}\grave{P} & \acute{S}\grave{P} \\ \acute{P}\grave{S} & \acute{S}\grave{S} \end{pmatrix}$$

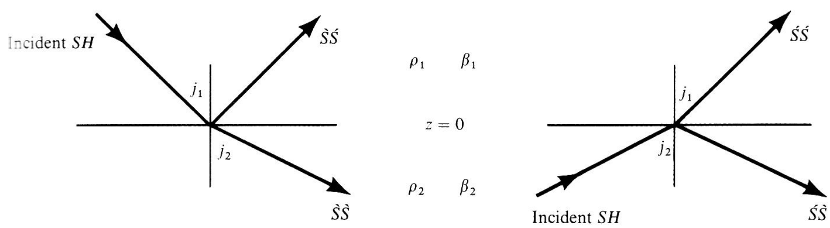
the scattering matrix of free surface reflections.

P-SV Reflection at free surface



Reflection and transmission of SH-waves

Reflection and transmission of SH-waves at solid-solid interface.



In case of downgoing incident SH:

$$\text{Incident SH: } \begin{pmatrix} 0 \\ S \\ 0 \end{pmatrix} e^{i\omega(px+\frac{cosj_1}{\beta_1}z-t)}$$

$$\text{Reflected SH: } \begin{pmatrix} 0 \\ S \\ 0 \end{pmatrix} \grave{S}\acute{S}e^{i\omega(px-\frac{cosj_1}{\beta_1}z-t)}$$

$$\text{Transmitted SH: } \begin{pmatrix} 0 \\ S \\ 0 \end{pmatrix} \grave{S}\grave{S}e^{i\omega(px+\frac{cosj_2}{\beta_2}z-t)}$$

Reflection and transmission of SH-waves

Continuity of displacement v and stress τ_{zy} at interface $z = 0$ yields:

$$\grave{S}\acute{S} = \frac{\rho_1\beta_1 \cos j_1 - \rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

$$\grave{S}\grave{S} = \frac{2\rho_1\beta_1 \cos j_1}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

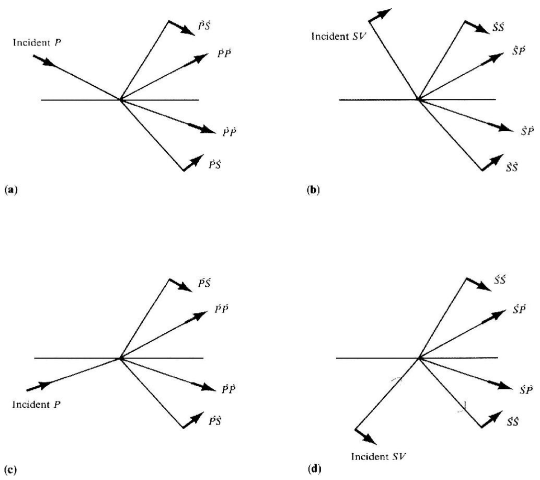
For upgoing incident SH we find:

$$\acute{S}\acute{S} = \frac{2\rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

$$\acute{S}\grave{S} = \frac{\rho_2\beta_2 \cos j_2 - \rho_1\beta_1 \cos j_1}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

Reflection and transmission of P-SV waves

Reflection and transmission of P- and SV-waves at solid-solid interface.



(a) Incident $\grave{P}$: $\begin{pmatrix} \sin i_1 \\ 0 \\ \cos i_1 \end{pmatrix} e^{i\omega(px + \frac{\cos i_1}{\alpha_1} z - t)}$

produces:

Reflection and transmission of P-SV waves

Reflected P: $\begin{pmatrix} \sin i_1 \\ 0 \\ -\cos i_1 \end{pmatrix} \grave{P}\acute{P} e^{i\omega(px - \frac{\cos i_1}{\alpha_1} z - t)}$

Reflected SV: $\begin{pmatrix} \cos j_1 \\ 0 \\ \sin j_1 \end{pmatrix} \grave{P}\acute{S} e^{i\omega(px - \frac{\cos j_1}{\beta_1} z - t)}$

Transmitted P: $\begin{pmatrix} \sin i_2 \\ 0 \\ \cos i_2 \end{pmatrix} \grave{P}\grave{P} e^{i\omega(px + \frac{\cos i_2}{\alpha_2} z - t)}$

Transmitted SV: $\begin{pmatrix} \cos j_2 \\ 0 \\ -\sin j_2 \end{pmatrix} \grave{P}\grave{S} e^{i\omega(px + \frac{\cos j_2}{\beta} z - t)}$

Boundary conditions on $z = 0$:
$u_x^1 = u_x^2$, $u_z^1 = u_z^2$, $\tau_{zx}^1 = \tau_{zx}^2$, $\tau_{zz}^1 = \tau_{zz}^2$.
4 equations with four unknows: $\grave{P}\acute{P}$, $\grave{P}\acute{S}$, $\grave{P}\grave{P}$, $\grave{P}\grave{S}$.
Similarly for incident $\grave{S}$, $\acute{P}$, and $\acute{S}$:
Scattering matrix has 4x4 elements.

Inhomogeneous waves

Inhomogeneous waves

What happens if the horizontal slowness becomes larger than 1/velocity of the medium?
Slowness vector of P-wave:

$$\bar{s}^P = \begin{pmatrix} \frac{\sin i}{\alpha} \\ 0 \\ \pm\frac{\cos i}{\alpha} \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ \pm\sqrt{\alpha^{-2} - p^2} \end{pmatrix}$$

If horizontal slowness $p > \frac{1}{\alpha}$: $\rightarrow s_z^P$ is imaginary,
$\rightarrow \sin i = \alpha p > 1 \rightarrow$ angle i is complex.
We speak of an inhomogeneous P-wave.
S-wave slowness:

$$\bar{s}^S = \begin{pmatrix} p \\ 0 \\ \pm\sqrt{\beta^{-2} - p^2} \end{pmatrix}$$

If $p > \frac{1}{\beta} \rightarrow s_z^S$ is imaginary $\rightarrow$ inhomogeneous S-wave.
(Then P-wave is inhomogeneous as well.)

Inhomogeneous waves

What does this imply? We had plane waves of type $\bar{u} \propto$:

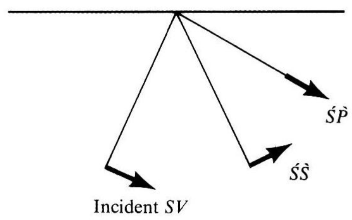
$$e^{i\omega(\bar{s}\cdot\bar{x}-t)} = e^{i\omega s_z z} e^{i\omega(px-t)}$$

s_z imaginary $\rightarrow s_z = \pm i|s_z| \rightarrow e^{i\omega s_z z} = e^{\mp\omega|s_z|z}$,
i.e. exponential decay or growth with depth.

We had $s^2 = c^{-2}$ where $c = \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$ or $c = \beta = \sqrt{\frac{\mu}{\rho}}$
For inhomogeneous waves we therefore have

- $p = s_x > \frac{1}{c} \rightarrow$ horizontal phase velocity smaller than underlying body wave speed.
- and $s_z^2 =$ negative $\rightarrow$ amplitudes decay exponentially from the interface.

Inhomogeneous waves: 1. SV-wave incident at free surface



Free surface incident SV-wave.
Reflected P-wave:

$$\bar{u}^{P,refl}(\bar{x},t) = S \begin{pmatrix} \sin i \\ 0 \\ \cos i \end{pmatrix} \acute{S}\grave{P}\ e^{i\omega \frac{\cos i}{\alpha} z}\ e^{i\omega(px-t)}$$

with

$$\acute{S}\grave{P} = \frac{4\frac{\beta}{\alpha}\ p\ \frac{\cos j}{\beta}\left(\frac{1}{\beta^2} - 2p^2\right)}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\ \frac{\cos i}{\alpha}\ \frac{\cos j}{\beta}}$$

and $\bar{s}^{P,refl} = (p, 0, \frac{\cos i}{\alpha}) = \left(p, 0, \sqrt{\alpha^{-2} - p^2}\right)$

Inhomogeneous waves: 1. SV-wave incident at free surface

$p < \frac{1}{\alpha}$: ordinary reflected P- and SV-waves.
$p = \frac{1}{\alpha}$: critical incidence for reflected P-wave.

Reflected P-wave for $\frac{1}{\alpha} < p < \frac{1}{\beta}$:

- s_z^P imaginary $\rightarrow$ inhomogeneous P-wave $\rightarrow$ P-wave decays with depth:

$$\omega > 0 : \frac{\cos i}{\alpha} = i\sqrt{p^2 - \alpha^{-2}}; \qquad \omega < 0 : \frac{\cos i}{\alpha} = -i\sqrt{p^2 - \alpha^{-2}}$$

- $\acute{S}\grave{P}$ no longer real $\rightarrow$ waveform changes w.r.t. incident wave due to phase shift.
- for positive ω:
  phase $u_x^{\acute{S}\grave{P}} \sim$ phase $(\acute{S}\grave{P})$
  phase $u_z^{\acute{S}\grave{P}} \sim$ phase $(\acute{S}\grave{P}) + \frac{\pi}{2}$
  $\rightarrow$ phase shift between horizontal and vertical component.

Inhomogeneous waves: 1. SV-wave incident at free surface

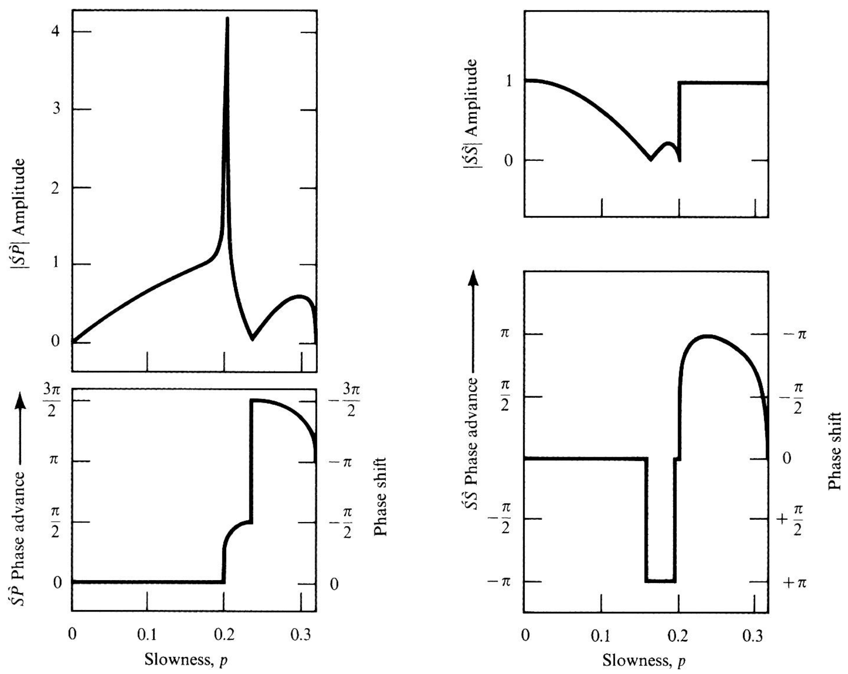
Reflected S-wave for $\frac{1}{\alpha} < p < \frac{1}{\beta}$:

- is not inhomogeneous,
- but changes waveform w.r.t. incident SV-wave, because $\acute{S}\grave{S}$ is complex:

$$\acute{S}\grave{S} = \frac{\left(\frac{1}{\beta^2} - 2p^2\right)^2 - 4p^2 \frac{\cos i}{\alpha} \frac{\cos j}{\beta}}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2 \frac{\cos i}{\alpha} \frac{\cos j}{\beta}}$$

Note $|\acute{S}\grave{S}| = 1$.

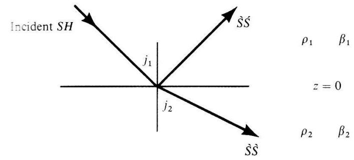
Inhomogeneous waves: 1. SV-wave incident at free surface



Amplitude and phase as function of p of $\acute{S}\grave{P}$ and $\acute{S}\grave{S}$ for incident SV-wave at free surface for model with α=5 km/s and β=3 km/s.

Inhomogeneous waves: 2. SH-wave incident at interface

Incident SH-wave at interface.



Transmitted SH-wave is inhomogeneous when $\frac{1}{\beta_2} < p < \frac{1}{\beta_1}$:
$\bar{u}^{\grave{S}\grave{S}} \propto e^{i\omega \frac{\cos j_2}{\beta_2} z} \; e^{i\omega(px-t)}$
It decays with depth: $\frac{\cos j_2}{\beta_2} = i\sqrt{p^2 - \beta_2^{-2}}$ (for $\omega > 0$).

Reflected SH-wave is not inhomogeneous for $\frac{1}{\beta_2} < p < \frac{1}{\beta_1}$, but with

$$\grave{S}\acute{S} = \frac{\rho_1\beta_1 \cos j_1 - \rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

we find total internal reflection ($|\grave{S}\acute{S}| = 1$) with phase shift.
(Critical angle of incidence when $p = \frac{1}{\beta_2}$, i.e. $j_1 = \sin^{-1}\frac{\beta_1}{\beta_2}$.)

Inhomogeneous waves: 3. Rayleigh and Stonely waves

What happens if incident and reflected waves are inhomogeneous?
At free surface: all P-SV waves are inhomogeneous if $p > \frac{1}{\beta} > \frac{1}{\alpha}$.
$\rightarrow$ Energy is not travelling toward and away from boundary, but channelled along the surface.
Inhomogeneous P-wave:

$$\grave{P} \begin{pmatrix} \alpha p \\ 0 \\ i\sqrt{\alpha^2 p^2 - 1} \end{pmatrix} e^{-\omega\sqrt{p^2 - \alpha^{-2}} z} \; e^{i\omega(px-t)}$$

Inhomogeneous SV-wave:

$$\grave{S} \begin{pmatrix} i\sqrt{\beta^2 p^2 - 1} \\ 0 \\ -\beta p \end{pmatrix} e^{-\omega\sqrt{p^2 - \beta^{-2}} z} \; e^{i\omega(px-t)}$$

$\grave{P}$ and $\grave{S}$: amplitudes at $z = 0$.

Inhomogeneous waves: 3. Rayleigh and Stonely waves

These two waves are coupled by boundary conditions $\tau_{zx} = \tau_{zz} = 0$ on $z = 0$.
$\tau_{zx} = 0$:

$$2p\alpha\beta i\sqrt{p^2 - \alpha^{-2}}\ \grave{P} + (1 - 2\beta^2 p^2)\grave{S} = 0$$

$\tau_{zz} = 0$:

$$(1 - 2\beta^2 p^2)\grave{P} - \frac{2\beta^3 pi}{\alpha}\sqrt{p^2 - \beta^{-2}}\ \grave{S} = 0$$

2 equations with 2 unknowns: non-trivial solution ($\grave{P} = \grave{S} = 0$) when determinant $= 0$:

$$\mathcal{R}(p) = (\beta^{-2} - 2p^2)^2 + 4p^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta} = 0$$

has one solution for p slightly larger than β^{-1}.

Inhomogeneous waves: 3. Rayleigh and Stonely waves

$\rightarrow$ It is possible to have a coupled pair of inhomogeneous P-SV-waves that propagates along the surface of a half-space: Rayleigh wave.
Features:

- $c_R = 1/p$ is slightly smaller than β
- c_R is independent of frequency (for half-space)
- elliptical particle motion

Stonely wave is an interface wave for two homogeneous half-spaces: decays upward and downward from interface.