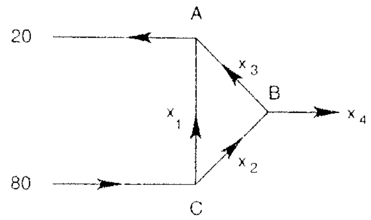
Extra problems Linear algebra and vector analysis

1. Find the interpolating polynomial $y(x) = a_0 + a_1 x + a_2 x^2$
   for the data $(1, 12)$, $(2, 15)$, and $(3, 16)$.
   Compute x for $y(x) = 15$.

2. Find the general flow pattern of the network shown in the figue. Assume that the total flow into the network equals the total outflow, and that the total flow into a junction equals the total flow out of the junction.
   Assuming that the flows are all nonnegative, what is the largest possible value of x_3?

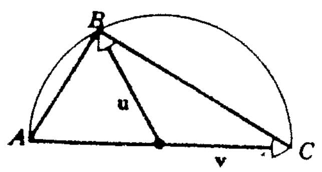


3. Use Cramer's rule to solve for x' and y' in terms of x and y.

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \end{aligned}$$

4. Let $\mathrm{p} = (2, k)$ and $\mathrm{q} = (3, 5)$. Find k such that
   (a) p and q are parallel.
   (b) p and q are orthogonal.
   (c) the angle between p and q is $\pi/3$.
   (d) the angle between p and q is $\pi/4$.

5. Use vector methods to prove that a triangle that is inscribed in a circle so it has a diameter for a side must be a right triangle. [*Hint.* Express the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in the figure in terms of u and v.



6. Prove: if a, b, c, and d lie in the same plane, then $(\mathrm{a} \times \mathrm{b}) \times (\mathrm{c} \times \mathrm{d}) = 0$.

7. Let A and B be square matrices of the same size. Is $(AB)^2 = A^2B^2$ a valid matrix identity? Justify your answer.

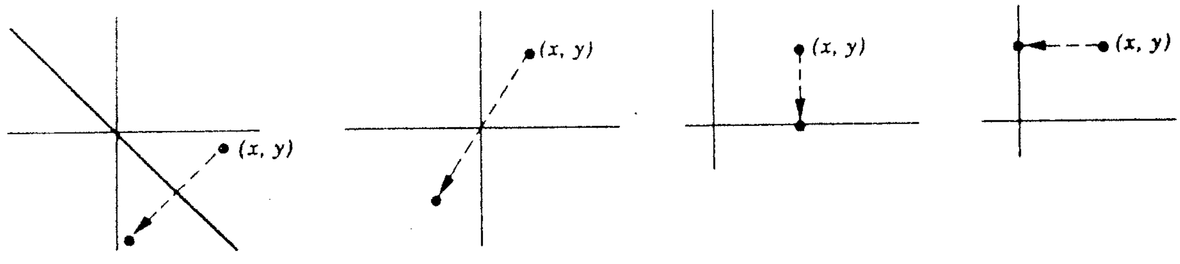
8. Indicate whether the statement is always true or sometimes false. Justify each answer by giving a logical argument or a counterexample.
(a) $\det(2A) = 2\det(A)$
(b) $|A^2| = |A|^2$
(c) $\det(I + A) = 1 + \det(A)$

9. How many 3×3 matrices A can you find such that

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ 0 \end{pmatrix}$$

10. Let O denote a 2×2 matrix, each of whose entries is zero.
(a) Is there a 2×2 matrix A such that $A \neq O$ and $AA = O$? Justify your answer.
(b) Is there a 2×2 matrix A such that $A \neq O$ and $AA = A$? Justify your answer.

11. Find the standard matrix for the plane linear transformation that maps a point (x, y) into
(a) its reflection about the line $y = -x$
(b) its reflection through the origin
(c) its orthogonal projection on the x-axis
(d) its orthogonal projection on the y-axis



(a) (b) (c) (d)

12. Show that multiplication by

$$A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

maps every point in the plane onto the line $y = 2x$.

13. Prove: If λ is an eigenvalue of an invertible matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1}, and $\mathbf{x}$ is a corresponding eigenvector.

14. Prove: If λ is an eigenvalue of matrix A, $\mathbf{x}$ is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$, and $\mathbf{x}$ is a corresponding eigenvector.

Exercises polar coordinates

15. Plot the following points and find the rectangular coordinates.

    (a) $(3,\ \pi/4)$
    (b) $(5,\ 2\pi/3)$
    (c) $(1,\ \pi/2)$
    (d) $(4,\ 7\pi/6)$
    (e) $(2,\ 4\pi/3)$
    (f) $(0,\ \pi)$

16. The following points are given in rectangular coordinates. Express the points in polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$.

    (a) $(-5,\ 0)$
    (b) $(2\sqrt{3},\ -2)$
    (c) $(0,\ -2)$
    (d) $(-8,\ -8)$
    (e) $(-3,\ 3\sqrt{3})$
    (f) $(1,\ 1)$

17. Identify the curve by transforming to rectangular coordinates.

    (a) $r = 2$
    (b) $r \sin\theta = 4$
    (c) $r = 3\cos\theta$

18. Prove that the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2) is:
    $d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$

Exercises cylindrical and spherical coordinates

19. The following points are given in cylindrical coordinates. Express each in rectangular coordinates and spherical coordinates.

    (a) $(1,\ \pi/4,\ 1)$
    (b) $(0,\ \pi/4,\ 10)$
    (c) $(1,\ -\pi/6,\ 0)$
    (d) $(2,\ 3\pi/4,\ -2)$

20. Describe the geometric meaning of the following mappings in spherical coordinates:

    (a) $(\rho,\ \theta,\ \phi) \rightarrow (\rho,\ \theta,\ \phi + \pi)$
    (b) $(\rho,\ \theta,\ \phi) \rightarrow (\rho,\ \pi - \theta,\ \phi)$

21. Express the following surfaces in spherical coordinates:

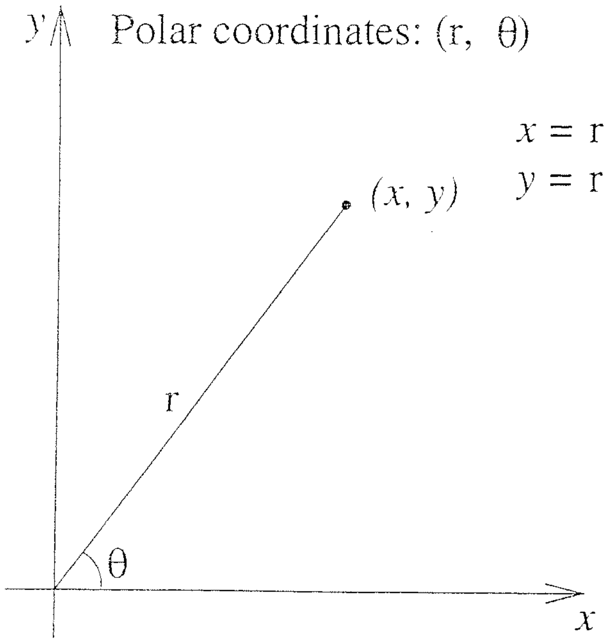
    (a) $xz = 1$
    (b) $x^2 + y^2 - z^2 = 1$

Exercises line integrals

22. Find the arc length s of the curve $y = 2x$ from $(1, 2)$ to $(2, 4)$ using the three expressions for ds of equation (3.1) in chapter 5 of the book.

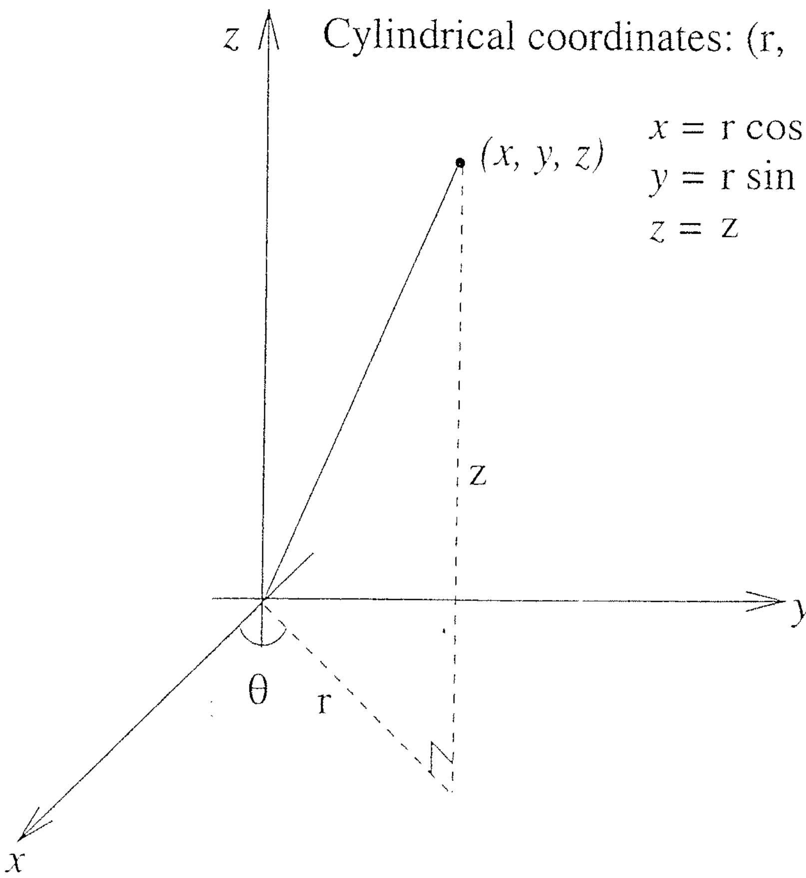
23. Find the arc length of the curve $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$. (Answer: $[(85)^{3/2} - 8]/243$)

Polar coordinates: (r, θ)



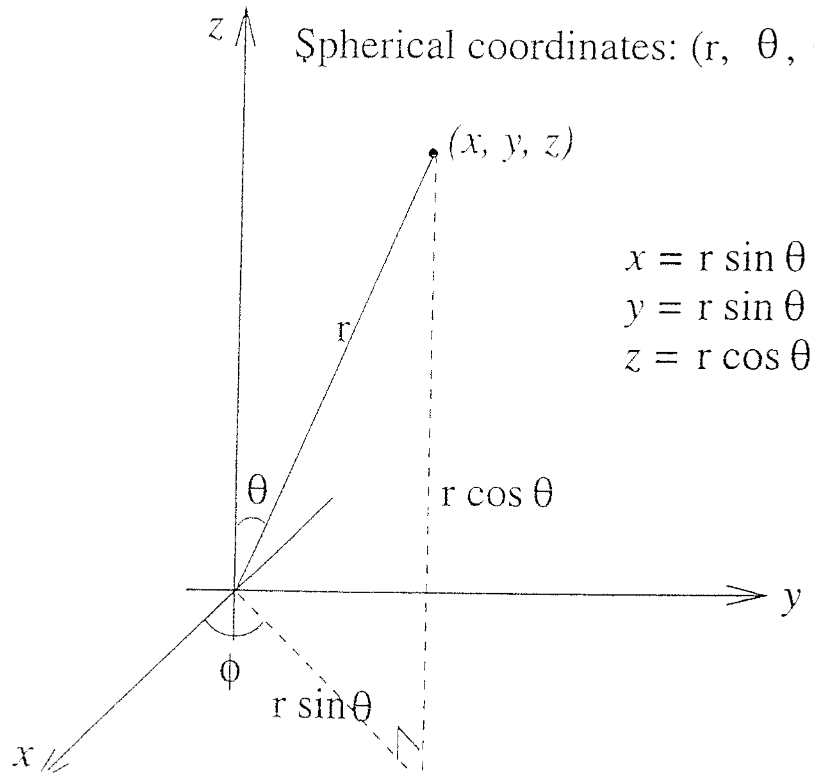
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Cylindrical coordinates: (r, θ, z)



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Spherical coordinates: (r, θ, ϕ)



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Problem: dating of rock samples using radioactive decay I.

The discovery of radioactivity has given geology an accurate tool for obtaining the age of rock samples. Using the decay of radioactive elements, important geologic events (mountain building) can be dated and, interestingly, the age of the Earth itself can be determined.

The principle underlying radiometric dating is the decay of a radioactive parent element P into a stable (i.e., non-radioactive) daughter element D. We will sketch the method using the decay of ^{87}Rb into ^{87}Sr. For this decay we can write (see text books on radioactive dating)

$$[^{87}\mathrm{Sr}]_{now} = [^{87}\mathrm{Sr}]_o + [^{87}\mathrm{Rb}]_{now}\,(\,e^{\lambda t} - 1) \tag{1}$$

where [ ] denotes the concentration of an element, t is the time (in the past) at which the rock sample was formed (or crystallized) and λ is the decay constant of ^{87}Rb. For this decay scheme, the problem arises that ^{87}Sr occurs naturally, so the concentration at the time of formation of the rock sample ($[^{87}\mathrm{Sr}]_o$) must be included in the calculations. For example, if we measure the concentration of these isotopes for a number of minerals, we must know $[^{87}\mathrm{Sr}]_o$ for every mineral separately, as there is no reason to assume that all minerals in a rock sample had the same $[^{87}\mathrm{Sr}]_o$ at the time of crystallization. A stable isotope of ^{87}Sr, that is not the result of any decay series is ^{86}Sr. If we measure its concentration, we can use that correct the concentration of ^{87}Sr. We may assume that whatever the concentration $[^{87}\mathrm{Sr}]_o$ was at the time of crystallization, the ratio $[^{87}\mathrm{Sr}]_o/[^{86}\mathrm{Sr}]_o$ was constant for all minerals in a rock sample. Equation (1) then becomes

$$\frac{[^{87}\mathrm{Sr}]_{now}}{[^{86}\mathrm{Sr}]_{now}} = \frac{[^{87}\mathrm{Sr}]_o}{[^{86}\mathrm{Sr}]_{now}} + \frac{[^{87}\mathrm{Rb}]_{now}}{[^{86}\mathrm{Sr}]_{now}}\,(\,e^{\lambda t} - 1)$$

$$= \frac{[^{87}\mathrm{Sr}]_o}{[^{86}\mathrm{Sr}]_o} + \frac{[^{87}\mathrm{Rb}]_{now}}{[^{86}\mathrm{Sr}]_{now}}\,(\,e^{\lambda t} - 1) \tag{2}$$

If we write $y = \dfrac{[^{87}\mathrm{Sr}]_{now}}{[^{86}\mathrm{Sr}]_{now}}$ and $x = \dfrac{[^{87}\mathrm{Rb}]_{now}}{[^{86}\mathrm{Sr}]_{now}}$, equation (2) becomes

$$y = ax + b \tag{3}$$

This is the equation of a straight line with slope $a = (\,e^{\lambda t} - 1)$ (from which the age t can be determined), and offset $b = \dfrac{[^{87}\mathrm{Sr}]_o}{[^{86}\mathrm{Sr}]_o}$.

If for a number of minerals the ratios $[^{87}\mathrm{Sr}]_{now}/[^{86}\mathrm{Sr}]_{now}$ (=y) and $[^{87}\mathrm{Rb}]_{now}/[^{86}\mathrm{Sr}]_{now}$ (=x) are measured and plotted against each other, the age of the sample t follows from the slope of the resulting line (which equals $(\,e^{\lambda t} - 1)$). In this way $[^{87}\mathrm{Sr}]_0$ does not have to be known for each mineral separately.

An important assumption in methods as these is that since time t, the rock has not undergone any chemical change, i.e. that none of the elements involved in the dating process have been added to or removed from the rock: it is assumed that the rock formed a 'closed system' since its formation, or crystallization.

a. For a given rock sample two sets of measurements of the ratios $[^{87}\mathrm{Sr}]_{now}/[^{86}\mathrm{Sr}]_{now}$ and $[^{87}\mathrm{Rb}]_{now}/[^{86}\mathrm{Sr}]_{now}$ were made:

	x	y
1	1.00	0.75
2	3.00	0.86

Determine the age t of this sample. The decay constant for ^{87}Rb is $\lambda = 1.42 \cdot 10^{-11}$ yr^{-1}. This sample was taken from one of the oldest rocks known, in western Greenland. Find also the increase in the isotope ratio $[^{87}\mathrm{Sr}]/[^{86}\mathrm{Sr}]$ between the time of crystallization and now for the two minerals. For this you have to find $b = \dfrac{[^{87}\mathrm{Sr}]_o}{[^{86}\mathrm{Sr}]_o}$. If the rock has been a closed system, this increase is entirely due to the decay process.

Problem: average density of the Earth's mantle and core.

In this problem we will try to obtain an estimate of the average densities of the Earth's mantle and core from only a few data from seismology and satellite orbits. In this way we can get an idea of how high the densities are deep in the Earth.

For a satellite orbiting the Earth, the time it takes to complete one orbit gives an accurate estimate of the total mass M_e of the Earth: $M_e = 5.92 \cdot 10^{24}$ kg. Therefore, any model describing the distribution of density in the Earth must satisfy the following equation

$$M_e = 4\pi \int_0^R \rho(r) r^2 dr \tag{1}$$

which gives the total mass M_e of an Earth model with density $\rho(r)$, which depends only on the distance r to the centre of the Earth. R is the radius of the Earth, $R = 6371$ km. In this problem we will only consider densities that depend on the distance (r) to the centre only.

a. Compute the average density of the Earth. (Compute (1) with $\rho(r) = \bar{\rho}$, a constant.) The answer should be given in units of kg/m^3.

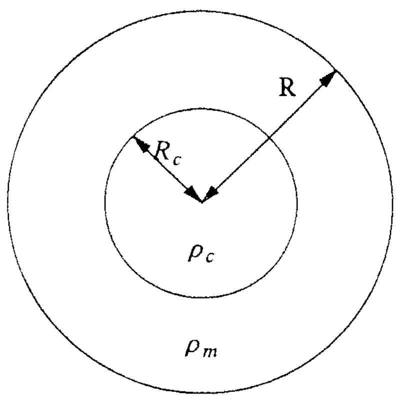
The principal moment of inertia of the Earth can be derived from a combination of observations of the gravity field and the precession of the Earth's axis of rotation. This moment of inertia describes how difficult it is to slow down or speed up the rotation of a body. It is a linear function of the density, so we can use it to constrain the density inside the Earth. The moment of inertia C is defined as mass times the square of the distance to the axis of rotation (r'):

$$C = \int_V r'^2 \, dm \tag{2}$$

We can use the fact that in our model of average densities of mantle and core the density depends only on distance to the Earth's centre and work out (2), to obtain

$$C = \frac{8}{3}\pi \int_0^R \rho(r)\, r^4 \, dr \tag{3}$$

where r is now the distance to the Earth's centre. (See the figure for a sketch of the different variables that are used in these equations.) For the Earth, $C = 8.0378 \cdot 10^{37}$ kg m^2.



Definition of the variables used in this problem.

The last bit of information we need comes from seismology. In 1906, the seismologist Oldham discovered the

Earth's (molten) core from an analysis of seismic waves reflected by its boundary with the Earth's (solid) mantle. The radius of the core R_c is now accurately known from seismological data: R_c = 3486 km. This is a little over half the Earth's radius.

We now have two equations with two unknowns (the average densities of mantle and core).

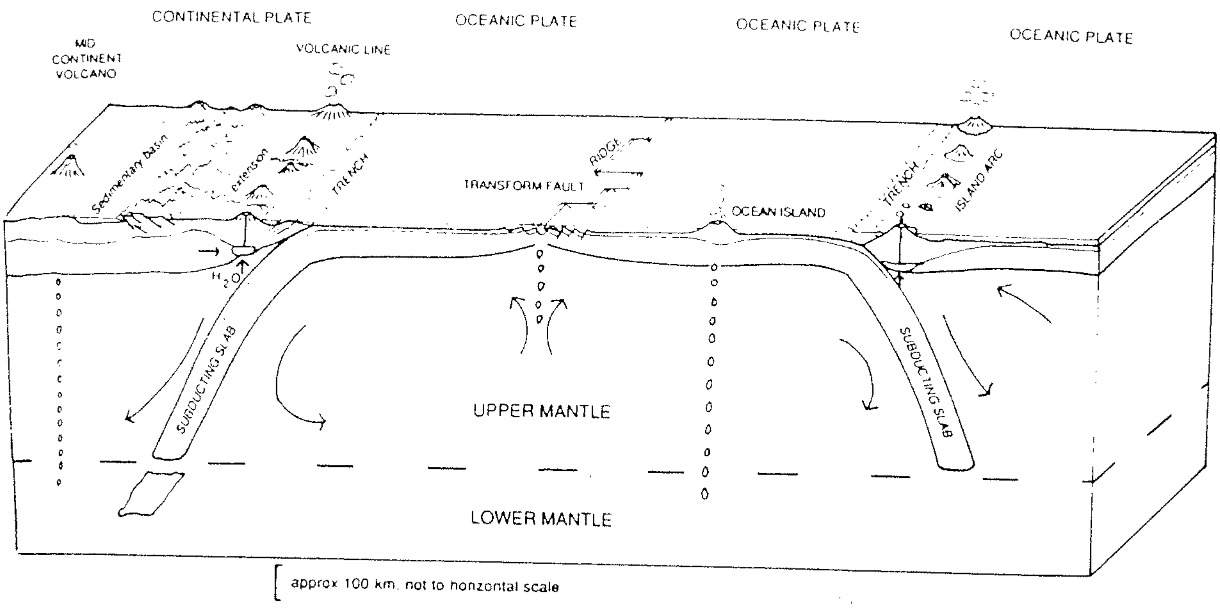
b. Assuming that the Earth consists of a homogeneous mantle and core (i.e. with constant density) only, write out (1) and (3) into two equations with two unknowns, the average densities of mantle (ρ_m) and core (ρ_c). (Split the integrations in (1) and (3) into two parts.) Verify that these equations are

$$\frac{4}{3}\pi\rho_c R_c^3 + \frac{4}{3}\pi\rho_m(R^3 - R_C^3) = M_e \qquad (4)$$

$$\frac{8}{15}\pi\rho_c R_c^5 + \frac{8}{15}\pi\rho_m(R^5 - R_c^5) = C$$

c. Now solve equations (4) for ρ_m and ρ_c. Work out the equations as far as possible, and fill in the numerical values given above only as the last step in your calculations.

Problem: rising and melting mantle material.



A sketch of the process of the creation of new oceanic crust at mid-oceanic ridges and the subduction of the oceanic crust at subduction zones. This causes volcanic activity above the subduction zone.

The strongest earthquakes occur near subduction zones, where oceanic crust disappears into the mantle. This process of destruction of crustal material implies that there must also be places where new crust is formed (see figure). This occurs at mid-oceanic ridges, where hot, molten mantle material rises up to form new basaltic oceanic crust, which then moves away from the ridge (the ridges are therefore called 'spreading ridges'). This hot material comes from considerable depths in the mantle. However, it melts only relatively close to the surface. This is because the melting temperature of the rock depends on pressure. At higher pressures it is more difficult to melt the material. Thus, as the mantle material rises (to good approximation with constant temperature), the pressure decreases, and at some depth the melt temperature is reached. Above that depth the material starts melting.

Consider a blob of hot mantle material that is rising up below a mid-oceanic ridge. Its temperature is 1800 °C. The pressure dependence of the melting point for this material is given by

$$T_m(p) = 1700 + 0.12 \cdot 10^{-6}\, p$$

where T_m is the melting temperature and p the pressure. We know the pressure increase with depth: $p(z) = \rho g z$, where ρ is the density of the material, g the gravitational acceleration and z the depth.

a. We now have two equations for the depth z and the pressure p, at which this blob start melting. Show that these two equations can be written in matrix form,

$$A\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

and that the matrix A has the form

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & -\rho g \end{pmatrix}$$

In this matrix, $\alpha = 0.12 \cdot 10^{-6}$.

b. Find the inverse of matrix A, by sweeping.

c. At what depth and pressure does this blob start melting? Use $\rho = 3300$ kg/m^3 and $g = 10$ m/s^2.

Problem: Deformation I.

The information about the crustal deformation on a large scale can be determined from deformation of rocks on a small scale. The deformation of rocks can be determined from cracks and faults in the rock, but also from fossils. The figure shows some deformed fossils. Because we know what the real fossils look like (from undeformed rock samples), we can deduce how much this rock has been deformed. In this problem we will look at some simple deformations and what they do to a unit square. A matrix notation can be used to describe deformations. We will see how multiple deformations can be treated, using three examples of simple deformations.

A simple deformation is called *simple shear*. Simple shear can be compared to the deformation of a card deck, with the bottom fixed and the top moved sideways. Therefore, it changes the shape of objects. Its matrix representation is

$$M_s = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \tag{1}$$

Pure shear is another form of a simple deformation. The result of pure shear is that the material is squeezed in one direction, while it expands in another direction. This deformation also changes the shape of objects. Its representation in matrix form is

$$M_p = \begin{pmatrix} \gamma & 0 \\ 0 & 1/\gamma \end{pmatrix} \tag{2}$$

The third example of a simple deformation is the *rotation*. This deformation does not change the shape, only the orientation of objects. Its deformation matrix is simply

$$M_r = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{3}$$

a. Calculate and draw the deformation caused by these three deformations on the unit square. Use the fact that the first column of a matrix represents the image of the first unit vector $\hat{x}$ after deformation; the second column represents the image of the second unit vector $\hat{y}$ after deformation. Use $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$ and $\alpha = 45°$. What do γ, β and α represent?

One can imagine that during its long history, a geologic formation can undergo multiple stages of deformation. The final deformation is then the result of all previous deformations. In matrix notation, this can be written as the product of the individual deformation matrices.

b. Starting from the 'unit' square with four corner points $p_i = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}$, compute $M_p M_s\, p_i$. Note that the notation $M_p M_s$ means that first M_s should be applied and then M_p. Do the same for $M_s M_p\, p_i$. Do you obtain the same result?

c. Now compute the matrix products $M_r M_s$, $M_s M_r$, $M_r M_p$ and $M_p M_r$ to convince yourself that these deformations do not commute, i.e. the order in which they are applied cannot be changed without altering the final result. This knowledge can be used in unwrapping the deformation history of a given rock.

Problem: Deformation II.

Consider the general deformation matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, e.g., a combination of rotation, pure shear, etc. It projects the coordinate system (x, y) on to the system (x', y'):

$$x' = Ax + By$$
$$y' = Cx + Dy$$

In this problem we will show that such a deformation, however complex, does not curve space, i.e. parallel structures in the undeformed medium remain parallel in the deformed medium. A line in the undeformed medium is given by $y = mx + n$. A series of parallel lines is obtained by varying the number n.

a. Compute the inverse of M, M^{-1}, to express both x and y in terms of x' and y'.

b. Find the line after deformation took place. Show that the expression for the deformed line obtained in a is again of the form $y' = ax' + b$, a straight line. Show that the slope of this new line does not depend on n. This proves that the series of parallel lines in the undeformed medium is also a series of parallel lines in the deformed medium.

Problem: Deformation III.

Two deformed trilobites.

In this problem we will derive a criterion for deformation matrices that conserve area (in two dimensions) or volume (in three dimensions). For simplicity we will only consider two dimensional space here. For this we will compute the deformation of the unit square by a general deformation matrix of the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

a. Compute $M\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $M\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and draw the result. As the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ obviously is unaffected by M, the result of these calculations are the sides of the deformed square. In a previous problem we have shown that parallel lines remain parallel after deformation. Therefore, the other two sides of the deformed square are parallel to the sides computed in a. Compute and draw $M\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to convince yourself that this is indeed the case.

b. The area of the deformed square (which is a parallelogram) is given by the length of the two sides, multiplied with the sine of the angle between these sides. From Lagrange's identity we know that this is equal to the length of the cross product of the two deformed sides of the parallelogram. Therefore, we can write, with $\vec{a}$ and $\vec{b}$ the two sides of the parallelogram

$$Area = \|\vec{a}\|\,\|\vec{b}\|\sin\theta = \|\vec{a}\times\vec{b}\|$$

Verify that this area is equal to the determinant of the matrix M (your answer will contain the numbers A, B, C and D). The criterion for zero area change is then $\det M = 1$.
A deformation that satisfies this restriction is said to induce zero *dilatation*. Dilatation e is defined as $A'/A = 1 + e$, where A and A' are the area of an object before and after the deformation, respectively.

Problem: Deformation IV.

Consider the deformation matrix

$$M = \begin{pmatrix} 9/4 & 7/4 \\ 7/4 & 9/4 \end{pmatrix}$$

It represents the deformation due to some combination of pure shear, simple shear, etc.

a. Draw the square with corner points $\begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}$, after it has been deformed by M.

b. Compute the eigenvalues and eigenvectors of M.

c. What do you think do the eigenvalues and eigenvectors mean for this deformation?

Problem: Deformation V.

A solid body rotation is the rotation induced by some deformation. For example, a pure shear does not induce a solid body rotation, a rotation, does, of course. Any general deformation matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written as the product of a rotation matrix and a deformation matrix that does not involve a rotation. A rotation over an angle α can be written in matrix form as $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

A deformation matrix that does not involve a rotation must have $b = c$, which can be understood from the rotation matrix: only if $\alpha = 0$ do we have $\sin(\alpha) = -\sin(\alpha)$ or $b = c$.

a. Assume that the deformation M given above induces a rotation over an angle α. Applying the deformation M, and then rotating back over an angle $-\alpha$ should result in a deformation with no solid body rotation. Compute the solid body rotation α induced by the deformation matrix M. For this, find the matrix product $R(-\alpha)M$ and impose the restriction that the resulting deformation matrix has equal off-diagonal elements ($b = c$). This gives an expression for the angle α, which is the solid body rotation induced by M. Your answer should contain the numbers a, b, c and d.