Lineaire Algebra en Vector Analyse

12. Interpretatie gradiënt, divergentie, rotatie

Hanneke Paulssen

Universiteit Utrecht

Samenvatting

- Interpretatie gradiënt, divergentie, rotatie
- Flux
- Laplaciaan, andere combinaties van gradiënt, divergentie, rotatie
- Lijnintegraal vectorveld
- Conservatief vectorveld en scalaire potentiaal

Gradiënt, divergentie, rotatie

Gradiënt scalarveld ϕ

$$\nabla\phi = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \phi = \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{pmatrix}$$

Divergentie vectorveld $\vec{V}$

$$\nabla \cdot \vec{V} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Rotatie (curl) vectorveld $\vec{V}$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

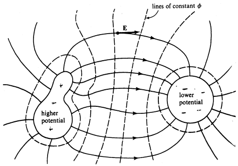
$$= \hat{\imath}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Interpretatie gradiënt

Gradiënt van scalarveld is vectorveld met richting en grootte van maximale afgeleide in elk punt.

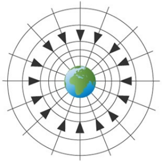
Voorbeelden scalar- ⇔ vectorvelden

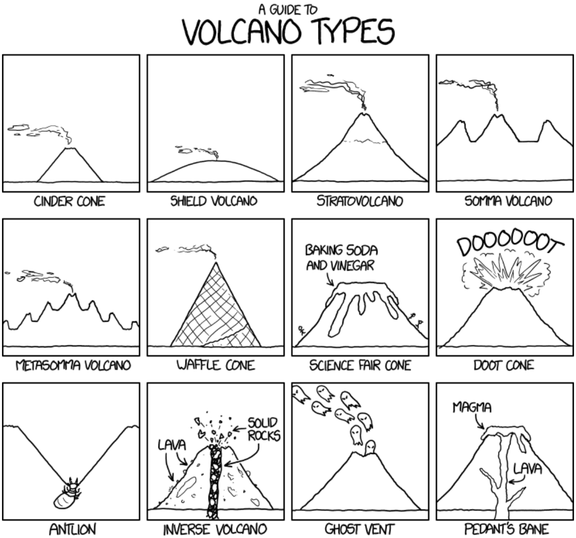
- $\vec{E} = -\nabla\phi$
  $\vec{E}$ = elektrische veld [V/m]
  ϕ = elektrische potentiaal [V]



- $\vec{q} = -k\nabla T$
  $\vec{q}$ = warmtestroom [W/m^2]
  T = temperatuur [K]
  k = thermische conductiviteit [W/mK]

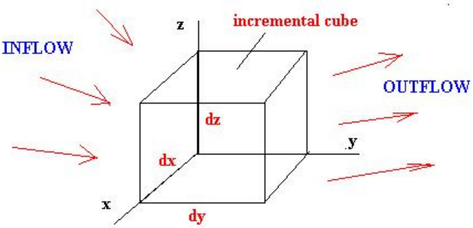
- Zwaartekracht ⇔ zwaartekrachtspotentiaal




A GUIDE TO
VOLCANO TYPES
CINDER CONE
SHIELD VOLCANO
STRATOVOLCANO
SOMMA VOLCANO
METASOMMA VOLCANO
WAFFLE CONE
BAKING SODA AND VINEGAR
SCIENCE FAIR CONE
DOOOOOOT
DOOT CONE
ANTLION
SOLID ROCKS
LAVA
INVERSE VOLCANO
GHOST VENT
MAGMA
LAVA
PEDANT'S BANE

Interpretatie divergentie

Vectorveld $\vec{v}$ ("stroming") door volume-element



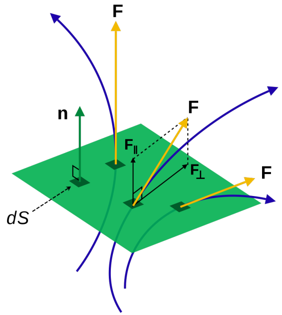
$$\vec{v} = \begin{pmatrix} v_x(x,y,z) \\ v_y(x,y,z) \\ v_z(x,y,z) \end{pmatrix}$$

Wat is de netto toe(af)name door volume-element?

Flux

Flux: netto toe(af)name vectorveld door oppervlak

Alleen component van vectorveld loodrecht op oppervlak geeft toe(af)name.



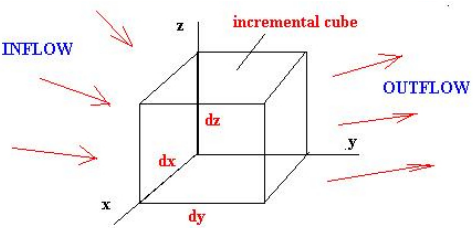
$\vec{F}$: vectorveld

$\hat{n}$: (naar buiten gerichte) normaal van oppervlakte-element dS

$\vec{F} \cdot \hat{n}$: component van $\vec{F}$ loodrecht op dS.

$\vec{F} \cdot \hat{n}\, dS$: Flux door oppervlakte-element dS

Interpretatie divergentie



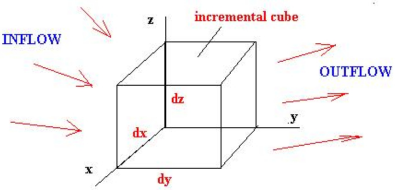
Bepaal eerst flux van $\vec{v}$ door volume-element in y-richting.

Linker zijvlak: normaal $\hat{n} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ oppervlak $\mathrm{dS} = dxdz$

Flux door linker zijvlak:

$$\vec{v} \cdot \hat{n}\, dS = \begin{pmatrix} v_x(x,y,z) \\ v_y(x,y,z) \\ v_z(x,y,z) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dxdz = -v_y(x,y,z)\, dxdz$$

Interpretatie divergentie



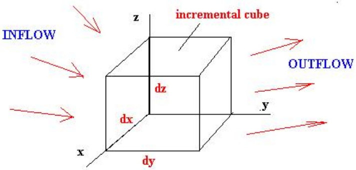
Rechter zijvlak: normaal $\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ oppervlak dS $= dxdz$

Flux rechter zijvlak: $\vec{v} \cdot \hat{n}\, dS = \begin{pmatrix} v_x(x, y+dy, z) \\ v_y(x, y+dy, z) \\ v_z(x, y+dy, z) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dxdz =$

$$v_y(x, y+dy, z)\, dxdz \approx \left(v_y(x, y, z) + \frac{\partial v_y}{\partial y} dy \right) dxdz$$

Analoog aan: $f(x+dx) \approx f(x) + \frac{df}{dx} dx$ (zie Intermezzo Lecture 11)

Interpretatie divergentie



Flux in y-richting:

Linker zijvlak: $-v_y(x,y,z)\,dxdz$

Rechter zijvlak: $\left(v_y(x,y,z) + \frac{\partial v_y}{\partial y}dy\right)dxdz$

Boven, onder, voor, achter: 0 (v_y geen loodrechte comp. op deze vlakken)

Totale flux in y-richting: $\frac{\partial v_y}{\partial y}dxdydz$

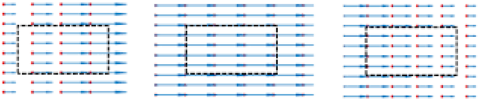
Analoog voor flux in x- en z-richting

Totale flux volume-element: $\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dxdydz =$

$\nabla \cdot \vec{v}$ maal volume-element

Interpretatie divergentie

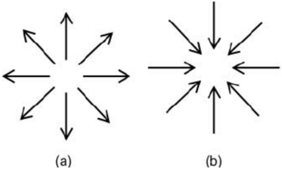
Divergentie is netto naar buiten gerichte flux per volume-element (volume-element $\to 0$). $\nabla \cdot \vec{v} = \text{div } \vec{v}$ is scalar veld.



Links: $\nabla \cdot \vec{v} > 0$

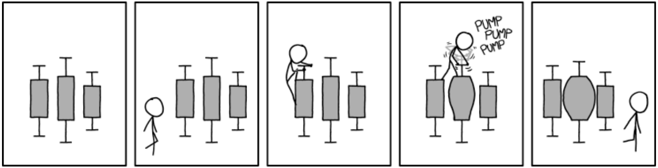
Midden: $\nabla \cdot \vec{v} = 0$

Rechts: $\nabla \cdot \vec{v} < 0$



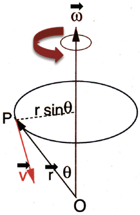
a: bron (source): divergentie positief

b: put (sink): divergentie negatief


PUMP
PUMP
PUMP

Rotatie

Rotatie (curl) is maat voor circulatie/draaiing van vectorveld.



Illustratie a.h.v. rotatie van punt

$\vec{v} = \vec{\omega} \times \vec{r}$

Kies coördinatenstelsel zo dat

$$\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix} \quad \text{en} \quad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega_z \\ x & y & z \end{vmatrix} = \hat{\imath}(-\omega_z y) + \hat{\jmath}(\omega_z x)$$

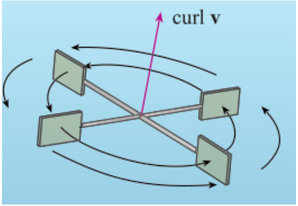
Rotatie van $\vec{v}$: $\nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega_z y & \omega_z x & 0 \end{vmatrix} = \hat{k}(2\omega_z)$

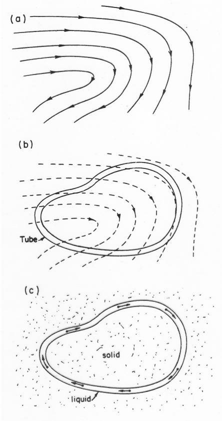
$$\nabla \times \vec{v} = 2\vec{\omega}$$

Rotatie

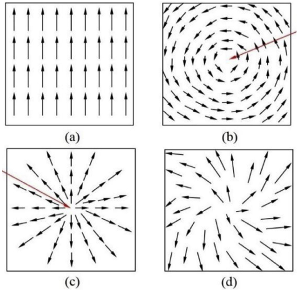
Rotatie is evenredig met hoeksnelheid, en is maat voor draaiing.

Rotatie geeft grootte van draaiing/circulatie en draaias.
Als $\nabla \times \vec{v}(\vec{r}) = \vec{0}$, dan geen draaiing van $\vec{v}$ in punt $\vec{r}$.

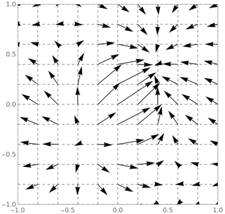




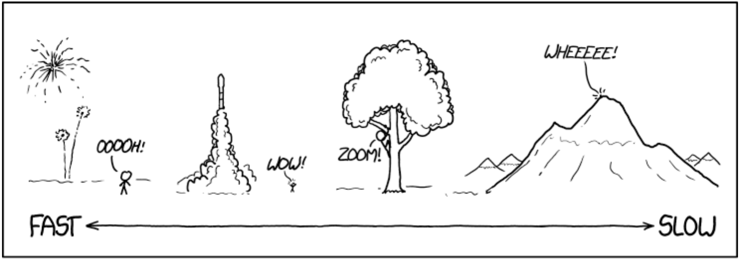
Divergentie en rotatie, voorbeelden



(a): $\nabla \cdot \vec{v} = 0, \ \ \nabla \times \vec{v} = \vec{0}$
(b): $\nabla \cdot \vec{v} = 0, \ \ \nabla \times \vec{v} \neq \vec{0}$
(c): $\nabla \cdot \vec{v} > 0, \ \ \nabla \times \vec{v} = \vec{0}$
(d): $\nabla \cdot \vec{v} \neq 0, \ \ \nabla \times \vec{v} \neq \vec{0}$



Bijna overal
$\nabla \cdot \vec{v} \neq 0, \ \ \nabla \times \vec{v} \neq \vec{0}$



MOST OF MY INTERESTS FALL UNDER "THINGS RISING UP FROM THE GROUND, HANGING IN THE AIR, AND THEN DRIFTING AWAY ON THE BREEZE," JUST ON VERY DIFFERENT TIMESCALES.

Combinaties gradiënt, divergentie, rotatie

- div grad ϕ:

$$\nabla \cdot \nabla \phi = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{pmatrix} = \boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi}$$

$\nabla^2\phi$ is Laplaciaan van ϕ

Laplaciaan van scalarveld is scalarveld.

- rot grad ϕ:

$$\nabla \times \nabla\phi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} =$$

$$\hat{\imath}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \hat{\jmath}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \vec{0}$$

Rotatie van gradiënt van scalarveld is altijd $\vec{0}$.

Combinaties gradiënt, divergentie, rotatie

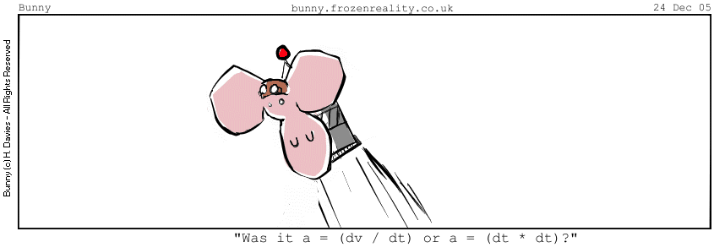
- div rot $\vec{v}$:
  $\nabla \cdot (\nabla \times \vec{v}) = 0$ Ga zelf na.
  Divergentie van rotatie van vectorveld is altijd 0.

- rot rot $\vec{v}$:
  $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - (\nabla \cdot \nabla)\vec{v} = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$
  ∇^2 is nu vector Laplaciaan

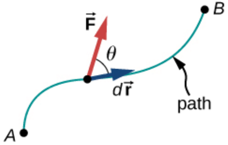
$$\nabla^2 \vec{v} = (\nabla \cdot \nabla)\vec{v} = \left[\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}\right]\vec{v} =$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{v}$$

$$= \begin{pmatrix} \partial^2 v_x/\partial x^2 + \partial^2 v_x/\partial y^2 + \partial^2 v_x/\partial z^2 \\ \partial^2 v_y/\partial x^2 + \partial^2 v_y/\partial y^2 + \partial^2 v_y/\partial z^2 \\ \partial^2 v_z/\partial x^2 + \partial^2 v_z/\partial y^2 + \partial^2 v_z/\partial z^2 \end{pmatrix} = \hat{\imath}\, \nabla^2 v_x + \hat{\jmath}\, \nabla^2 v_y + \hat{k}\, \nabla^2 v_z$$


Bunny
bunny.frozenreality.co.uk
24 Dec 05
Bunny (c) H. Davies - All Rights Reserved
"Was it a = (dv / dt) or a = (dt * dt)?"

Lijnintegraal vectorveld



Berekening arbeid langs pad:

$$dW = \vec{F} \cdot d\vec{r}$$

$$W = \int_{pad} \vec{F} \cdot d\vec{r}$$

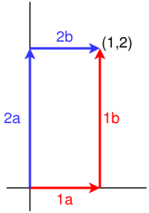
kracht: $\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$ padlengte-element: $d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$$

$$W = \int_{pad} F_x(x, y, z)dx + \int_{pad} F_y(x, y, z)dy + \int_{pad} F_z(x, y, z)dz$$

Lijnintegraal vectorveld, voorbeeld

$$\vec{F} = \begin{pmatrix} -y \\ x \end{pmatrix}$$



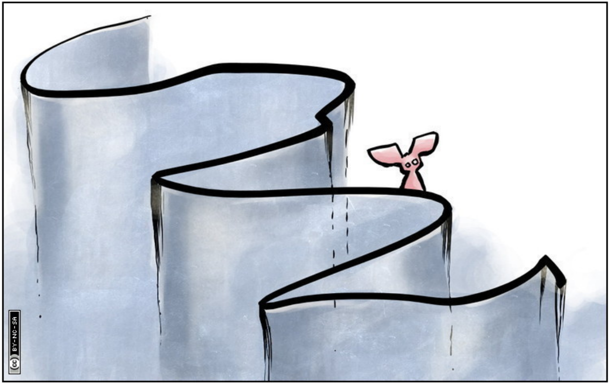
Arbeid van (0,0) naar (1,2)
via twee verschillende paden.

$$W = \int_{(0,0)}^{(1,2)} \vec{F} \cdot d\vec{r} = \int_{(0,0)}^{(1,2)} \begin{pmatrix} -y \\ x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_{(0,0)}^{(1,2)} -ydx + xdy$$

$$= \int_{x=0}^{1} -ydx + \int_{y=0}^{2} xdy$$

Pad 1: $W = \int_{x=0}^{1} \underbrace{0}_{y=0 \text{ op 1a}} dx + \int_{y=0}^{2} \underbrace{1}_{x=1 \text{ op 1b}} dy = 2$

Pad 2: $W = \int_{x=0}^{1} \underbrace{-2}_{y=2 \text{ op 2b}} dx + \int_{y=0}^{2} \underbrace{0}_{x=0 \text{ op 2a}} dy = -2$



The Bunny couldn't remember what they came down here to find.

Conservatief vectorveld

$\vec{F}$ is conservatief als $\int_A^B \vec{F} \cdot d\vec{r}$ onafhankelijk van integratiepad:

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B dW = W(B) - W(A)$$

Veel fysische vectorvelden zijn conservatief.

Als $\vec{F}$ conservatief, dan gesloten lijnintegraal (kringintegraal) nul:

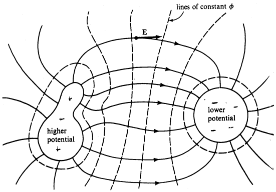
$$\oint \vec{F} \cdot d\vec{r} = 0$$

Als $\nabla \times \vec{F} = \vec{0}$ in alle punten van de ruimte, dan $\vec{F}$ conservatief.

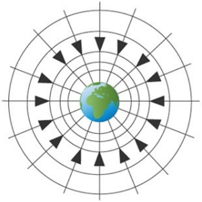
Conservatief vectorveld en scalaire potentiaal

Conservatief vectorveld $\vec{F}$ heeft scalaire potentiaal ϕ zodanig dat $\vec{F} = -\nabla\phi$

Bijvoorbeeld:



- $\vec{E} = -\nabla\phi$
  $\vec{E}$: elektrische veld
  ϕ: elektrische potentiaal



- $\vec{F}_{grav} = -\nabla\phi_{grav}$
  $\vec{F}_{grav} = m\vec{g}$: zwaartekracht ($\downarrow$)
  ϕ_{grav}: zwaartekrachtspotentiaal
  $d\phi_{grav} = mgdz$ ($dz \uparrow$) of

$$\vec{F}_{grav} = -\begin{pmatrix} \partial\phi_{grav}/\partial x \\ \partial\phi_{grav}/\partial y \\ \partial\phi_{grav}/\partial z \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}$$

Conservatief vectorveld en scalaire potentiaal

Hoe vind je scalaire potentiaal ϕ van convervatief vectorveld $\vec{F}$?
Door inverse operatie van ∇, d.w.z. integratie van $\vec{F}$.

Integratie langs pad geeft arbeid. Kies pad bijvoorbeeld:

$$W = \int_{(0,0,0)}^{(x,y,z)} \vec{F} \cdot d\vec{r} = \int_{(0,0,0)}^{(x,0,0)} F_x dx + \int_{(x,0,0)}^{(x,y,0)} F_y dy + \int_{(x,y,0)}^{(x,y,z)} F_z dz$$

$$\begin{aligned} \text{Arbeid} &= - \text{ verandering van potentiaal (energiebehoud)} \\ &= -(\phi - \phi_0) \end{aligned}$$

$W = -\phi$ als beginpotentiaal $\phi_0 = 0$, ofwel $\phi = -W$ als $\phi_0 = 0$

Controleer dat $\nabla\phi = -\vec{F}$ voor werkcollegeopgaven 8 en 11 van sectie 6.8.


done for today
ZZZZZ