Lineaire Algebra en Vector Analyse

10. Niet-cartesische coördinatenstelsels

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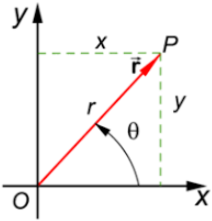
Samenvatting

- Poolcoördinaten
- Cilindercoördinaten
- Bolcoördinaten
- Jacobi matrix, Jacobiaan
- Eenheidsvectoren pool- en cilindercoordinaten

Poolcoördinaten



2D Poolcoördinaten: (r, θ)

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

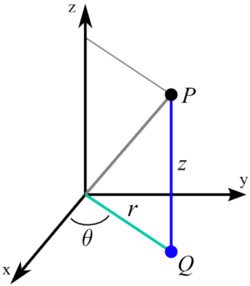
$$\theta = \arctan\left(\frac{y}{x}\right) \quad \text{voor } -\frac{\pi}{2} < \arctan\left(\frac{y}{x}\right) \leq \frac{\pi}{2}$$

Voor $0 \leq \theta < 2\pi$:

$$\begin{aligned}\theta &= \arctan\left(\frac{y}{x}\right) \quad \text{als } x \geq 0,\ y \geq 0 \quad \text{(1e kwadrant)}\\ &= \arctan\left(\frac{y}{x}\right) + \pi \quad \text{als } x < 0 \quad \text{(2e, 3e kwadrant)}\\ &= \arctan\left(\frac{y}{x}\right) + 2\pi \quad \text{als } x > 0\ y < 0 \quad \text{(4e kwadrant)}\end{aligned}$$

Maak figuur met locatie punt. Check of hoek in juiste kwadrant.

Cilindrische coördinaten



3D Cilindercoördinaten: (r, θ, z)

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

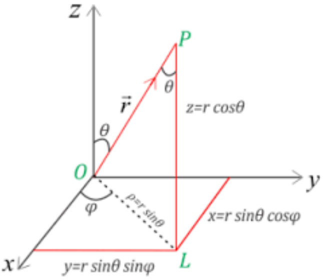
$$r = \sqrt{x^2 + y^2}$$

Voor $0 \leq \theta < 2\pi$:

$$\theta = \arctan\left(\frac{y}{x}\right) \quad \text{als } x \geq 0,\ y \geq 0 \quad \text{(1e kwadrant x-y vlak)}$$
$$= \arctan\left(\frac{y}{x}\right) + \pi \quad \text{als } x < 0 \quad \text{(2e, 3e kwadrant x-y vlak)}$$
$$= \arctan\left(\frac{y}{x}\right) + 2\pi \quad \text{als } x > 0\ y < 0 \quad \text{(4e kwadrant x-y vlak)}$$

Bolcoördinaten

3D Bolcoördinaten (spherical coordinates): (r, θ, ϕ)



$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad r \geq 0$$

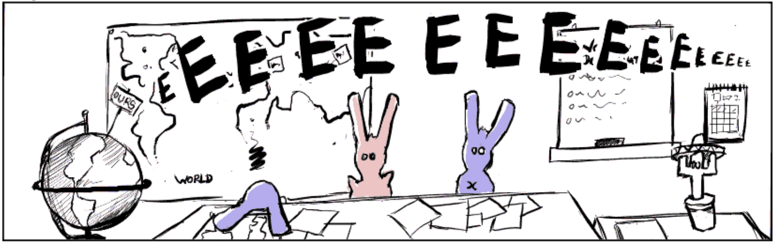
$$\theta = \arccos\left(\frac{z}{r}\right) \qquad 0 \leq \theta \leq \pi$$

$$\begin{aligned} \phi &= \arctan\left(\frac{y}{x}\right) \quad x \geq 0,\ y \geq 0 \\ &= \arctan\left(\frac{y}{x}\right) + \pi \quad x < 0 \\ &= \arctan\left(\frac{y}{x}\right) + 2\pi \quad x > 0,\ y < 0 \qquad 0 \leq \phi < 2\pi \end{aligned}$$

Let op 1: $r_{cilinder} \neq r_{bol}$ en $\theta_{cilinder} = \phi_{bol}$

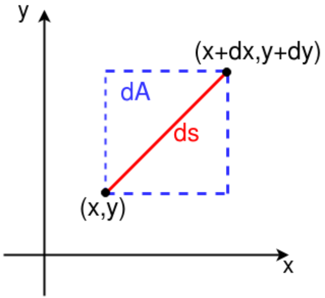
Let op 2: De conventie van θ en ϕ kan ook omgekeerd zijn.

Bunny 01 Nov 04



"The Bunnys realised they'd overlooked something terribly important..."

Lengte-, oppervlaktevlakte-element cartesische coörd. (2D)



Van (x, y) naar $(x + dx, y + dy)$

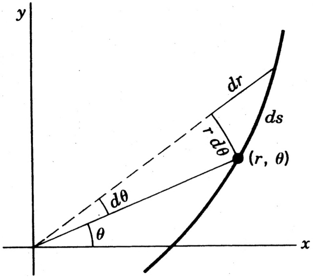
Lengte-element ds:

$(ds)^2 = (dx)^2 + (dy)^2$

Oppervlakte-element dA:

$dA = dxdy$

Lengte-element poolcoördinaten



Van (r, θ) naar $(r + dr, \theta + d\theta)$

Uit figuur:

$(ds)^2 = (dr)^2 + (rd\theta)^2$

Mathematisch:

$x = r\cos\theta$

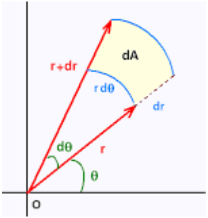
$y = r\sin\theta$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta$$

$$(ds)^2 = (dx)^2 + (dy)^2 = \ldots = (dr)^2 + (rd\theta)^2$$

Oppervlakte-element poolcoördinaten



Uit figuur:

$dA = dr\ rd\theta = r\ drd\theta$

Mathematisch:

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}}_{Jacobi\ matrix\ pool \rightarrow cart} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

Jacobiaan$_{pool \rightarrow cart}$: $J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$dA = |J|\ drd\theta = r\ drd\theta$

Lengte-, volume-element cartesische coördinaten (3D)

Van (x, y, z) naar $(x + dx, y + dy, z + dz)$

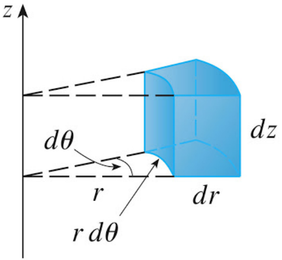
Lengte-element ds:

$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

Volume-element dV:

$dV = dxdydz$

Lengte-, volume-element cilindercoördinaten



Van (r, θ, z) naar $(r + dr, \theta + d\theta, z + dz)$

Lengte-element ds:
$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2$

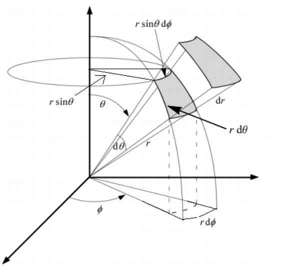
Volume-element dV:
$dV = dr\ rd\theta\ dz = r\ drd\theta dz$

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ dz \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Jacobi\ matrix\ cil \to cart} \begin{pmatrix} dr \\ d\theta \\ dz \end{pmatrix}$$

$$\text{Jacobiaan}_{cil\to cart}\text{:}\ J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$dV = |J|\ drd\theta dz = r\ drd\theta dz$$

Lengte-, volume-element bolcoördinaten



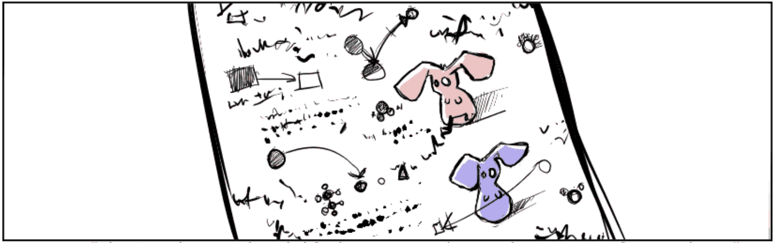
Van (r, θ, ϕ) naar $(r+dr, \theta+d\theta, \phi+d\phi)$

Lengte-element ds:
$(ds)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$

Volume-element dV:
$dV = dr\ rd\theta\ r\sin\theta d\phi = r^2\sin\theta\ drd\theta d\phi$

Jacobiaan$_{bol\rightarrow cart}$: $J = r^2\sin\theta$ (zie boek hoofdstuk 5 eq. (4.14))

$dV = |J|\ drd\theta d\phi = r^2\sin\theta\ drd\theta d\phi$



"The Bunnies wondered if there was going to be a test afterwards..."

Eenheidsvectoren, vectorelement cartesische coördinaten

Cartesisch coördinatenstelsel: eenheidvectoren $\hat{\imath}, \hat{\jmath}, \hat{k}$

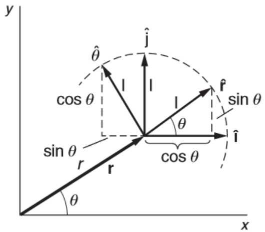
Vectorelement tussen $d\vec{s}$ tussen $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ en $\begin{pmatrix} x + dx \\ y + dy \\ z + dz \end{pmatrix}$:

$d\vec{s} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$

Lengte van $d\vec{s}$ (vector) is ds (scalar): $ds = |d\vec{s}\,|$

$(ds)^2 = d\vec{s} \cdot d\vec{s}$ (dot product)

Eenheidsvectoren in poolcoördinaten



Logische eenheidsvectoren voor een punt in poolcoördinaten zijn

- in radiële richting $\hat{e}_r$ (figuur: $\hat{r}$)
- in "θ" richting $\hat{e}_\theta$ (figuur: $\hat{\theta}$)

$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

Zijn $\hat{e}_r$ en $\hat{e}_\theta$ eenheidsvectoren?

Bijv: $\hat{e}_r \cdot \hat{e}_r = \cos^2\theta + \sin^2\theta = 1$ Ja

Ga zelf na dat $\hat{e}_r$ en $\hat{e}_\theta$ orthogonaal zijn (onderling loodrecht)

Let op: Richting van $\hat{e}_r$ en $\hat{e}_\theta$ varieert van punt tot punt!

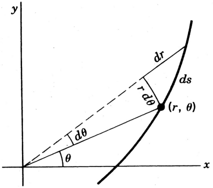
Vectorelement poolcoördinaten

Mathematische afleiding van $\hat{e}_r$ en $\hat{e}_\theta$ m.b.v. vectorelement $d\vec{s}$.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\begin{aligned}
d\vec{s} &= dx\,\hat{\imath} + dy\,\hat{\jmath} \\
&= \left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right)\hat{\imath} + \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)\hat{\jmath} \\
&= (\cos\theta dr - r\sin\theta d\theta)\,\hat{\imath} + (\sin\theta dr + r\cos\theta d\theta)\,\hat{\jmath} \\
&= dr\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + rd\theta\,(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \\
&= dr\,\hat{e}_r + rd\theta\,\hat{e}_\theta
\end{aligned}$$

$$(ds)^2 = d\vec{s}\cdot d\vec{s} = (dr)^2 + (rd\theta)^2$$



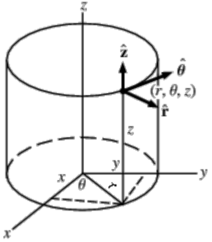
Eenheidsvectoren, vectorelement cilindercoördinaten

Eenheidsvectoren punt cilindercoördinaten

$\hat{\mathrm{e}}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ (figuur: $\hat{\mathrm{r}}$)

$\hat{\mathrm{e}}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ (figuur: $\hat{\theta}$)

$\hat{\mathrm{e}}_z = \hat{\mathrm{k}}$ (figuur: $\hat{z}$)



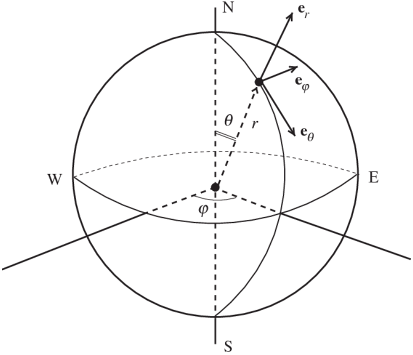
Vectorelement $d\vec{s}$:

$d\vec{s} = dr\,\hat{\mathrm{e}}_r + rd\theta\,\hat{\mathrm{e}}_\theta + dz\,\hat{\mathrm{e}}_z$

Lengte-element ds:

$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2$

Eenheidsvectoren in bolcoördinaten



$$\hat{e}_r = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

$$d\vec{s} = dr\,\hat{e}_r + rd\theta\,\hat{e}_\theta + r\sin\theta d\phi\,\hat{e}_\phi$$

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$$


KEEP
CALM
AND
THATS ALL FOR
TODAY