Extra Problems Linear Algebra and Vector Analysis

1. $a_0 = 7$, $a_1 = 6$, $a_2 = -1$
   solutions for $y(x) = 15$: $x = 2$ or $x = 4$

2. $x_{3,max} = 20$

3. $x' = x\cos\theta + y\sin\theta$
   $y' = y\cos\theta - x\sin\theta$

4. a) $k = \frac{10}{3}$
   b) $k = -\frac{6}{5}$
   c) $k = -0.03$ or $k = -3.6$
   d) $k = \frac{1}{2}$ or $k = -8$

5. -

6. -

7. -

8. a) not true (only true if A is 1×1 matrix)
   b) true
   c) not true

9. You can find one matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

10. a) Yes: for $x \in \mathbb{R}$

$$A = \begin{pmatrix} x & -x \\ x & -x \end{pmatrix}$$

b) Yes, for example unit matrix. Next conditions must be satisfied:

$$\begin{cases} a_{11}^2 + a_{12}a_{21} = a_{11} \\ a_{11}a_{12} + a_{12}a_{22} = a_{12} \\ a_{11}a_{21} + a_{21}a_{22} = a_{21} \\ a_{21}a_{12} + a_{22}^2 = a_{22} \end{cases}$$

11. a)

$$T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

b)

$$T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

c)

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

d)

$$T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

12. -

13. -

14. -

15. a) $(x, y) = (\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$
b) $(x, y) = (-\frac{5}{2}, \frac{5}{2}\sqrt{3})$
c) $(x, y) = (0, 1)$
d) $(x, y) = (-2\sqrt{3}, -2)$
e) $(x, y) = (-1, -\sqrt{3})$
f) $(x, y) = (0, 0)$

16. a) $(r, \theta) = (5, \pi)$
b) $(r, \theta) = (4, \frac{11\pi}{6})$
c) $(r, \theta) = (2, \frac{3\pi}{2})$
d) $(r, \theta) = (8\sqrt{2}, \frac{5\pi}{4})$
e) $(r, \theta) = (6, \frac{2\pi}{3})$
f) $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$

17. a) $\sqrt{x^2 + y^2} = 2$
b) $y = 4$
c) $x^2 + y^2 = 3x$

18. -

19. a) $(x, y, z) = (\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 1)$; $(r, \theta, \phi) = (\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4})$
b) $(x, y, z) = (0, 0, 10)$; $(r, \theta, \phi) = (10, 0, \frac{\pi}{4})$
c) $(x, y, z) = (\frac{1}{2}\sqrt{3}, -\frac{1}{2}, 0)$; $(r, \theta, \phi) = (1, \frac{\pi}{2}, -\frac{\pi}{6})$
d) $(x, y, z) = (-\sqrt{2}, \sqrt{2}, -2)$; $(r, \theta, \phi) = (2\sqrt{2}, \frac{3\pi}{4}, \frac{3\pi}{4})$

20. a) Rotation over an angle of π (180°) around the z-axis: $x \to -x$, $y \to -y$.
b) Reflection in $x - y$ plane: $z \to -z$.

21. a) $r^2 \cos\theta \sin\theta \cos\phi = 1$, or $r^2 \cos\phi \sin 2\theta = 2$
b) $r^2(\sin^2\theta - \cos^2\theta) = 1$, or $-r^2 \cos 2\theta = 1$

22. $ds = \sqrt{5}$

23. $[(85)^{3/2} - 8]/243$

Problem: Dating of rock samples using radioactive decay I.

$t = 3.7705\ Gyr$
$b = 0.695$
Increase in isotope ratio $[^{87}Sr]/[^{86}Sr]$ is 0.055 (for 1) and 0.165 (for 2)

Problem: Average density of the earth's mantle and core.

a) $\bar{\rho} = 5465.3\ kgm^{-3}$
b) -
c) $\rho_c = \frac{6M_e(R^5-R_c^5)-15C(R^3-R_c^3)}{8\pi(R_c^3R^5-R_c^5R^3)}$, $\rho_m = \frac{15C-6M_eR_c^2}{8\pi(R^5-R_c^2R^3)}$

Problem: Rising and melting mantle material.

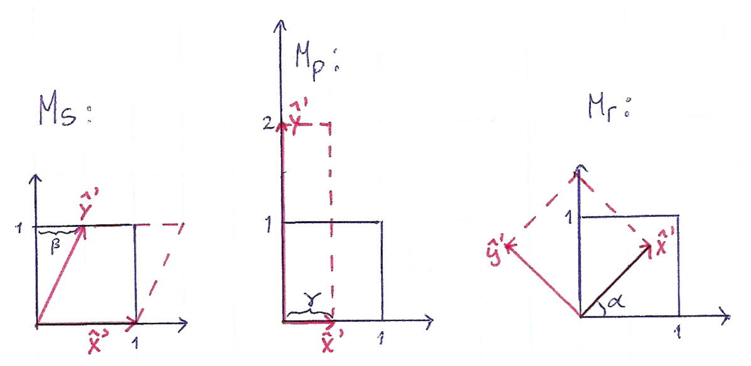
a) -
b)

$$A^{-1} = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ \frac{1}{\alpha\rho g} & -\frac{1}{\rho g} \end{pmatrix}$$

c) $p = 8.33 \times 10^2\ MPa$ and $z = 25.3\ km$

Problem: Deformation I.

a) α = angle of rotation, β = shear, γ = compression.



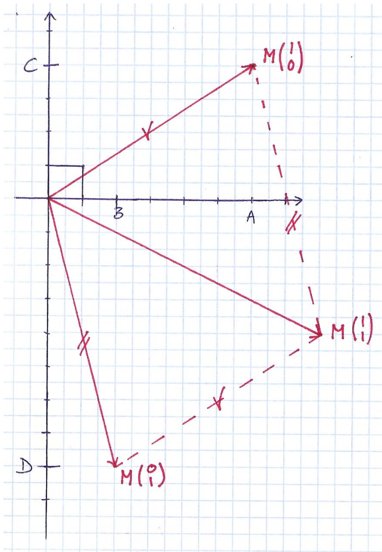
b) $M_pM_sp_i \neq M_sM_pp_i$
c) $M_rM_sp_i \neq M_sM_rp_i$, $M_rM_pp_i \neq M_pM_rp_i$

Problem: Deformation II.

a) $x = \frac{x'D-y'B}{AD-BC}$, $y = \frac{-x'C+y'A}{AD-BC}$
b) $y' = \frac{C+mD}{A+mB}x' + \frac{n}{A+mB}$

Problem: Deformation III.

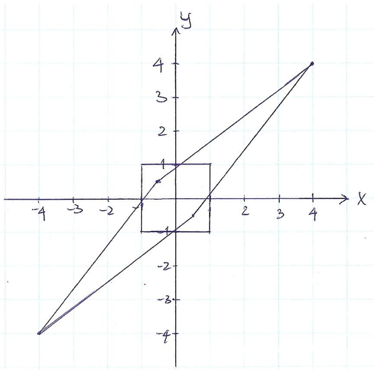
a)



b) -

Problem: Deformation IV.

a)



b) $\lambda_1 = \frac{1}{2}$: $\mathbf{r_1} = (1, -1)$, $\lambda_2 = 4$: $\mathbf{r_2} = (1, 1)$
c) -

Problem: Deformation V.

a) $\alpha = \arctan(\frac{c-b}{a+d})$

Solutions to Book Exercises (not (correctly) given in Boas)

Chapter 3

2.1)

$$\begin{cases} M_{11}x_1 + M_{12}x_2 + M_{13}x_3 = k_1 \\ M_{21}x_1 + M_{22}x_2 + M_{23}x_3 = k_2 \\ M_{31}x_1 + M_{32}x_2 + M_{33}x_3 = k_3 \end{cases}$$

2.2) Two equations, four unknowns:

$\sum_{j=1}^{4} M_{ij}x_j = k_i$, $i = 1, 2$

Four equations, two unknowns:

$\sum_{j=1}^{2} M_{ij}x_j = k_i$, $i = 1, 2, 3, 4$

2.10) Inconsistent, no solution

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.13) $x = -2$, $y = 2z + \frac{5}{2}$

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & 5/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

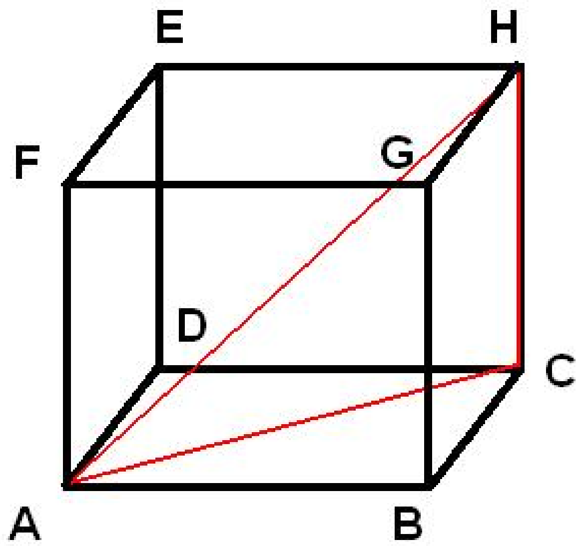
2.16) $R = 3$

3.3) 1

3.6) 4

3.15) Use equations from *problems*, not from text!
(2.3): $(x, y) = (-3, 5)$
(2.11): $(x, y, z) = (2, -1, -3)$

4.14) Example of space diagonal: AH; example of face diagonal: AC; example of edge: CH. Different results are possible ($\theta = 70.5°$, or $\theta = 109.5°$), but the sum is 180°.



4.22) Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos\gamma$, where $\gamma = 180° - \theta$

4.23) $\mathbf{A} \cdot \mathbf{B} = 0$: $\mathbf{B}$ doesn't have to be 0, *e.g.* $\mathbf{B} = (1, 1, 1)$.
$\mathbf{A} \times \mathbf{B} = \mathbf{0}$: $\mathbf{B}$ doesn't have to be 0.
$\mathbf{A} \cdot \mathbf{B} = 0$ and $\mathbf{A} \times \mathbf{B} = \mathbf{0}$: $\mathbf{B}$ must be 0.

4.24) Use Lagrange's identity:
$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$

5.37) Intersect at $(1, -3, 4)$.

6.5)

$$AA^T = \begin{pmatrix} 30 & -13 \\ -13 & 30 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 8 & 8 & 2 & 2 \\ 8 & 10 & 3 & -7 \\ 2 & 3 & 1 & -4 \\ 2 & -7 & -4 & 41 \end{pmatrix}$$

$$BB^T = \begin{pmatrix} 20 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 10 \end{pmatrix}$$

$$B^T B = \begin{pmatrix} 14 & 4 \\ 4 & 18 \end{pmatrix}$$

$$CC^T = \begin{pmatrix} 14 & 1 & 1 \\ 1 & 21 & -6 \\ 1 & -6 & 2 \end{pmatrix}$$

$$C^T C = \begin{pmatrix} 21 & -2 & -3 \\ -2 & 2 & 5 \\ -3 & 5 & 14 \end{pmatrix}$$

6.8) Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with a and b constant.
$5x^2 + 3y^2 = 30$

6.9) Singular if determinant is 0.

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 22 & 44 \\ -11 & -22 \end{pmatrix}$$

6.10)

$$AC = AD = \begin{pmatrix} 11 & 12 \\ 33 & 36 \end{pmatrix}$$

6.25) Use:
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

8.5) $b = \frac{(\mathbf{A}\times\mathbf{V})\cdot\hat{\mathbf{n}}}{(\mathbf{A}\times\mathbf{B})\cdot\hat{\mathbf{n}}}$
$a = \frac{(\mathbf{B}\times\mathbf{V})\cdot\hat{\mathbf{n}}}{(\mathbf{B}\times\mathbf{A})\cdot\hat{\mathbf{n}}}$

8.21)

$$\begin{vmatrix} x_1 & y_1 & z_1 & -1 \\ x_2 & y_2 & z_2 & -1 \\ x_3 & y_3 & z_3 & -1 \\ x_4 & y_4 & z_4 & -1 \end{vmatrix} = 0$$

8.23 $\lambda = 3$: $x = 2y$, $\mathbf{r} = (2,1)s$
$\lambda = 8$: $2x = -y$, $\mathbf{r} = (1,-2)t$

9.7) Only for symmetric, antisymmetric and orthogonal matrices.
Symmetric: $A_{ij} = A_{ji} = A^T_{ij}$
Antisymmetric: $A_{ij} = -A_{ji} = -A^T_{ij}$
Orthogonal: $A^{-1}_{ij} = A^T_{ij} = A_{ji}$

10.3**a)** Label the vectors **A**, **B**, **C**, **D**.
$\cos(\mathbf{AB}) = \frac{1}{\sqrt{15}}$
$\cos(\mathbf{AC}) = \frac{\sqrt{2}}{3}$
$\cos(\mathbf{BD}) = \frac{17}{\sqrt{345}}$
$\cos(\mathbf{AD}) = \frac{3}{\sqrt{23}}$
$\cos(\mathbf{BC}) = \frac{16}{3\sqrt{30}}$
$\cos(\mathbf{CD}) = \frac{20}{3\sqrt{46}}$

11.1) Use equations from *text*, not from problems!
$x_1 = \frac{1}{\sqrt{5}}, y_1 = \frac{2}{\sqrt{5}}, x_2 = \frac{-2}{\sqrt{5}}, y_2 = \frac{1}{\sqrt{5}}$, or
$x_1 = \frac{1}{\sqrt{5}}, y_1 = \frac{2}{\sqrt{5}}, x_2 = \frac{2}{\sqrt{5}}, y_2 = \frac{-1}{\sqrt{5}}$

11.4) Use equations from *text*, not from problems!

11.5) Use equations from *text*, not from problems!

Chapter 5

2.36) $\frac{2}{3}$; Wrong answer in book!!

4.1**a)** $A = a^2\pi$; b) $s = 2\pi a$

4.4**a)** $A = 4\pi a^2$; b) $V = \frac{4}{3}\pi a^3$

Chapter 6

3.1) $(\mathbf{A} \cdot \mathbf{B})\mathbf{C} = (0, 6, 6)$
$\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) = (-4, 2, 2)$
$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = -8$
$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -8$
$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (0, 4, -4)$
$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (-4, 0, -8)$

6.1) $(-16, -12, 8)$

10.5) $4\pi 5^5 = 12500\pi$; Wrong answer in book!!

11.2**a)** $4y\hat{k}$; b) -, c) $2ab^2$; $\iint (\nabla \times \mathbf{A}) \cdot d\bar{\sigma} = \iint (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, d\sigma$