Normal modes

Standing waves 1-D

We start from the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

And use separation of variables $u(x,t) = X(x)T(t)$
resulting in

$$\frac{1}{T}\frac{d^2 T}{dt^2} = \frac{c^2}{X}\frac{d^2 X}{dx^2} = -\omega^2$$

We solve for X and T separately, giving

$$T(t) = \left[\begin{array}{l}\cos(\omega t) \\ \sin(\omega t)\end{array}\right. \qquad X(x) = \left[\begin{array}{l}\cos(\omega x/c) \\ \sin(\omega x/c)\end{array}\right.$$

Standing waves 1-D

The general solution is

$$u(x,t) = C_1 \cos(\omega t)\cos(\omega x/c) + C_2 \cos(\omega t)\sin(\omega x/c) + C_3 \sin(\omega t)\cos(\omega x/c) + C_4 \sin(\omega t)\sin(\omega x/c)$$

We use the boundary conditions

(1) $u(0,t) = 0 \rightarrow C_1 = C_3 = 0$

(2) $u(L,t) = 0 \rightarrow \sin(\omega L/c) = 0$

$$\frac{\omega L}{c} = (n+1)\pi$$

$$\boxed{\omega_n = \frac{(n+1)\pi c}{L}}$$

Discrete eigenfrequencies

Standing waves 1-D

$$\omega_n = \frac{(n+1)\pi c}{L}$$

Discrete eigenfrequencies

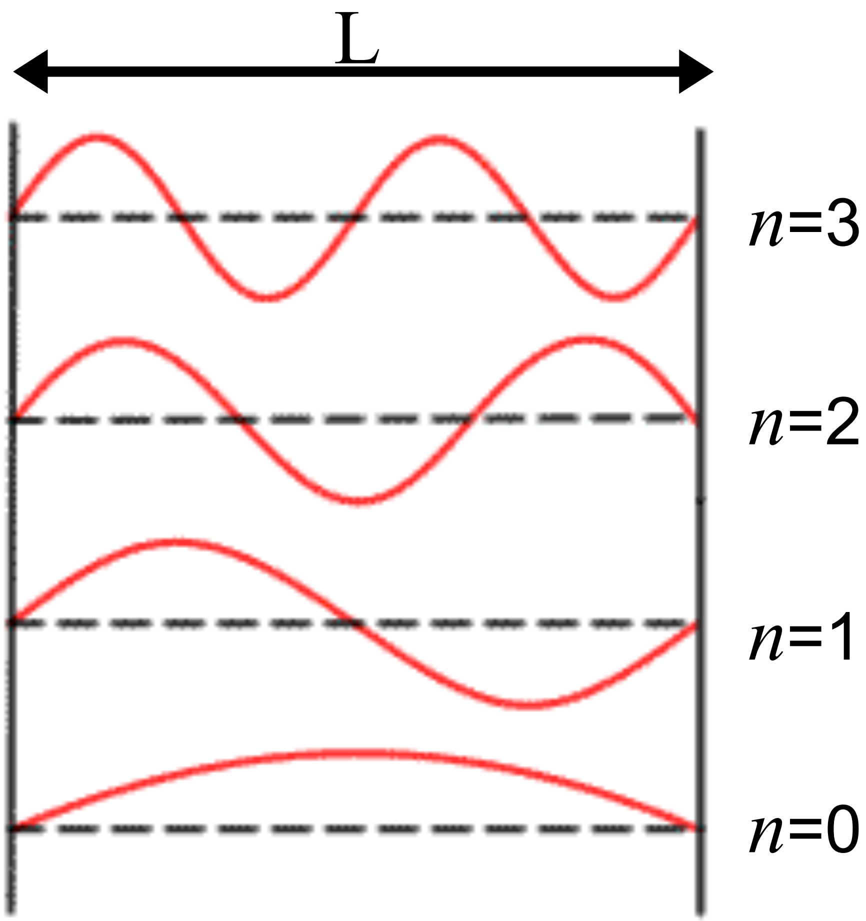
$$\sin(\frac{\omega_n x}{c})$$

Eigenfunctions or normal modes

Solutions are of the form

$$u(x,t) = \sum_{n=0}^{\infty} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \sin(\frac{\omega_n x}{c})$$

Standing waves 1-D



Standing waves on a string

Frequency of standing wave on a string:

$$\omega_n = \frac{(n+1)\pi c}{L}$$

where c = wave velocity

Fourier spectrum will have spikes at ω_n

Normal mode summation

The amplitudes A_n depend on the source

$$A_n = \sin(n\pi x_s/L)F(\omega_n)$$

So, the normal mode summation is

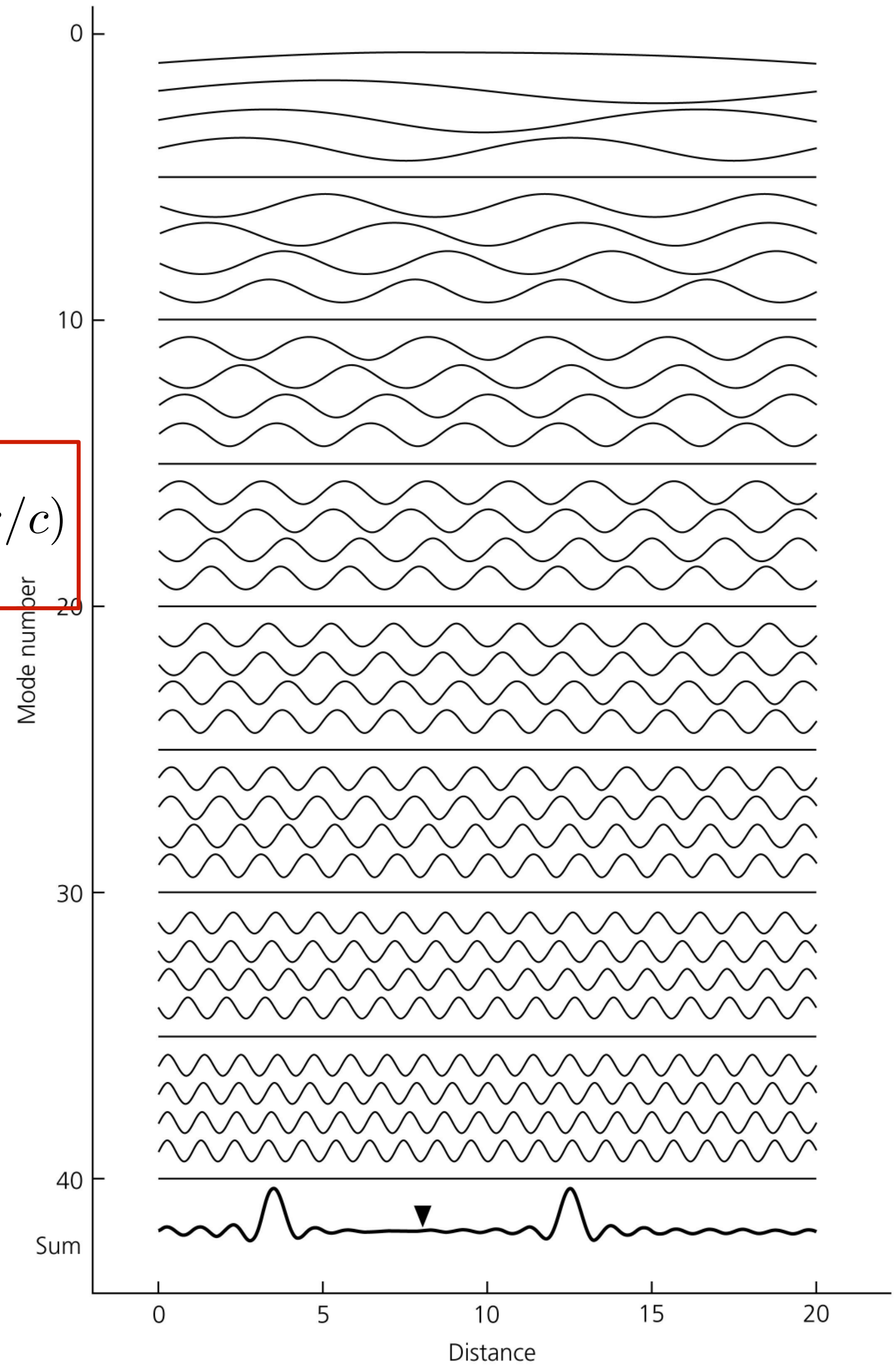
$$u(x,t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)F(\omega_n)\cos(\omega_n t)\sin(\omega_n x/c)$$

with the source, at x_s=8, described by

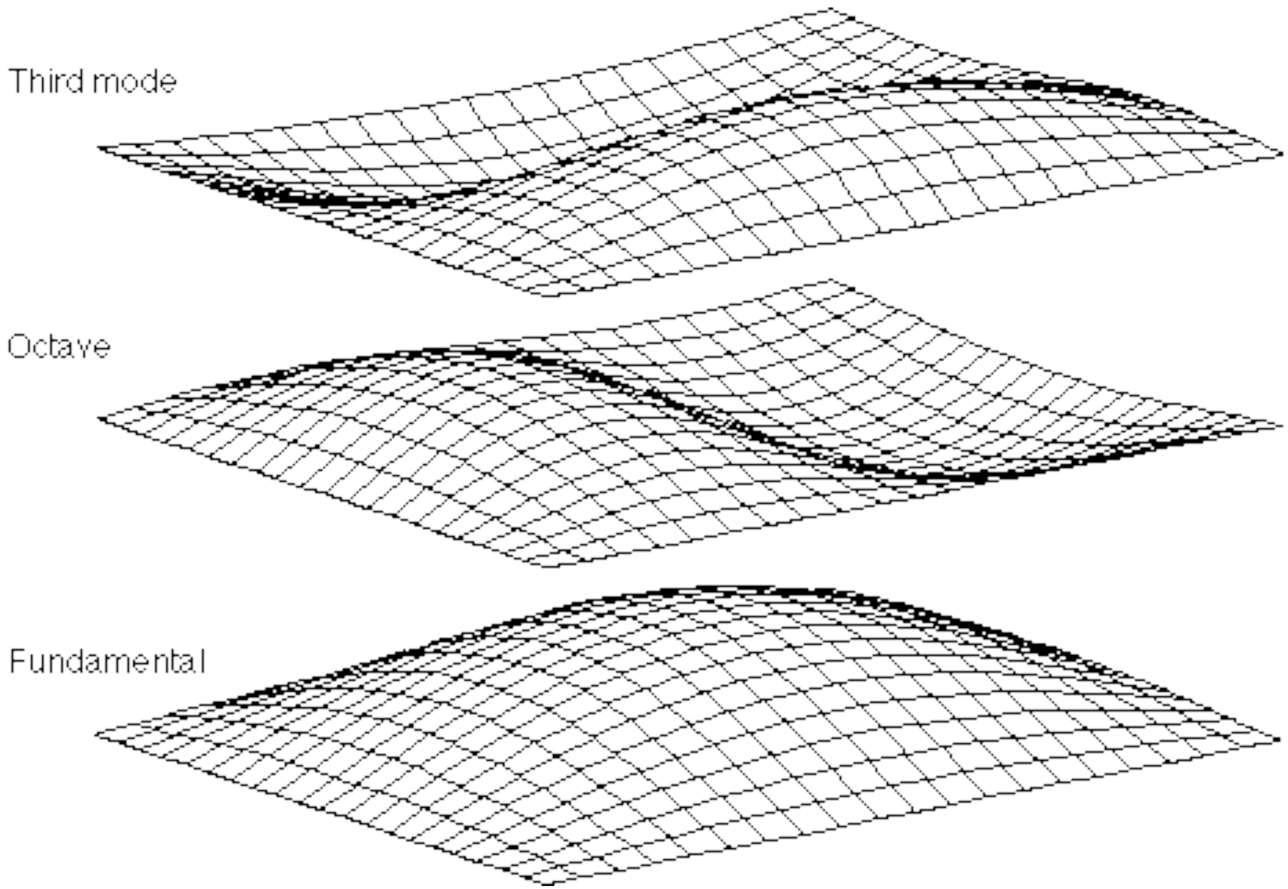
$$F(\omega_n) = \exp[-(\omega_n \tau)^2/4]$$

for $\tau = 0.2$

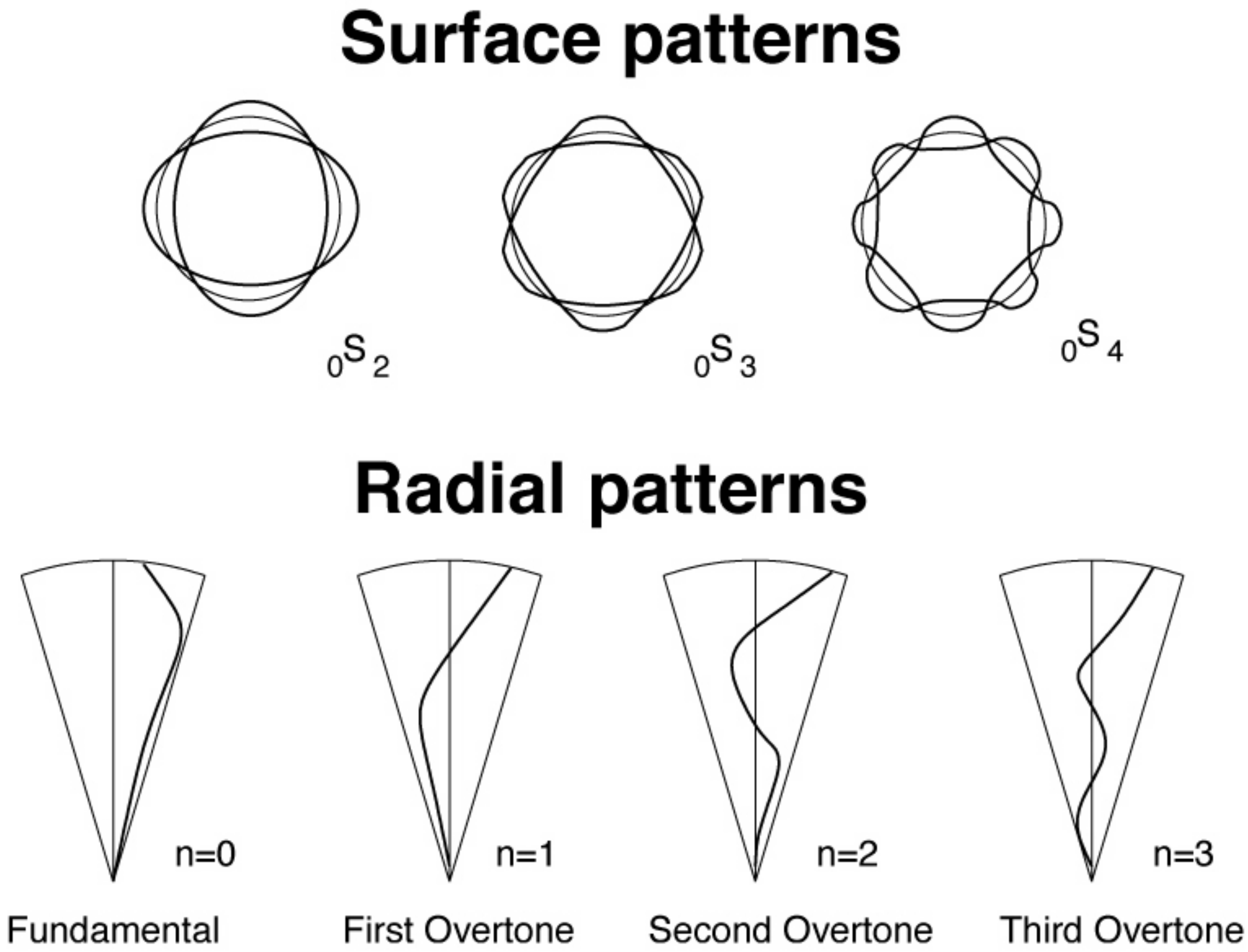
Figure 2.2-8: Waves on a string as a summation of modes.



Standing waves 2-D



Standing waves 3-D



Spherical harmonics

$Y_l^m(\theta,\phi)$

Angular order l

$l=0,1,2,\ldots$

Azimuthal order m

$m=-l,-l+1,1,..0,\ldots l$

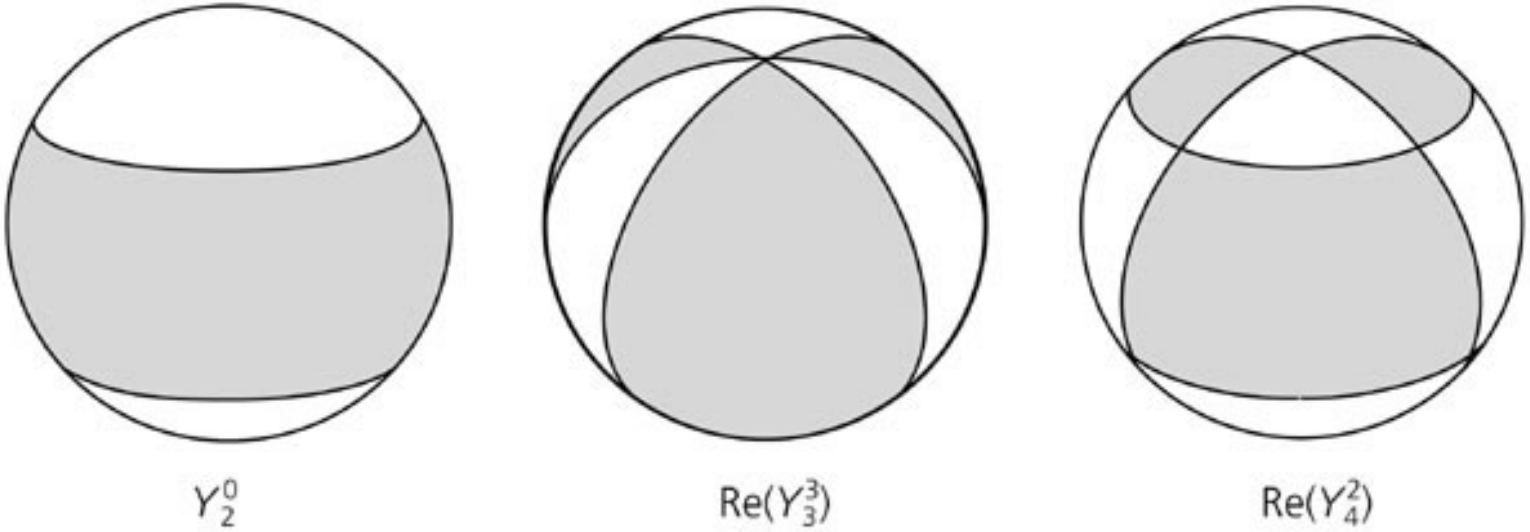
Spherical Bessel function

$j_n(kr)$

Radial order n

$n=0,1,2,\ldots$

Spherical harmonics



$$Y_l^m(\theta,\phi) = (-1)^m \left[\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

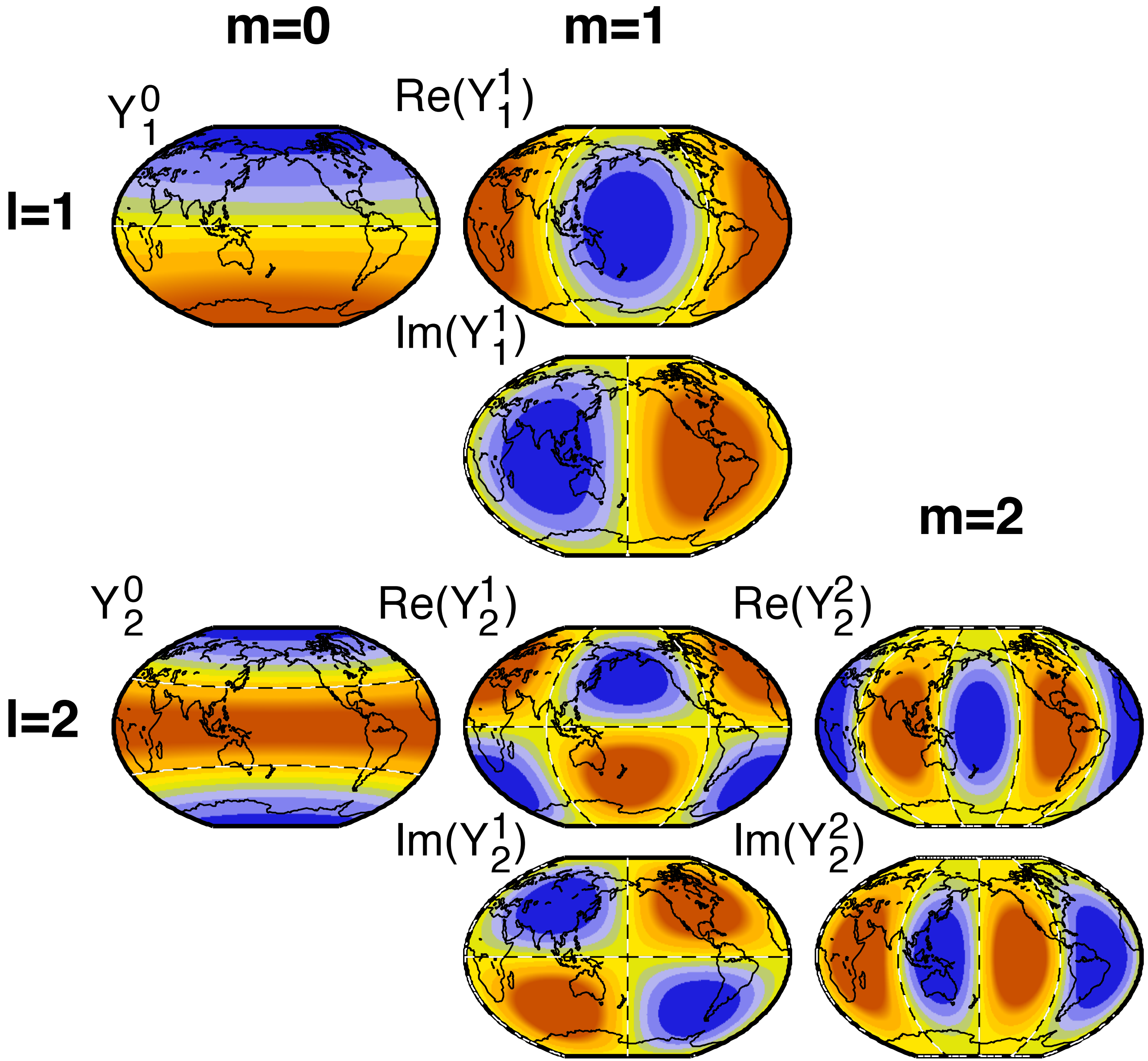
(where $P_l^m(\cos\,\theta)$ is associated Legendre function)

The angular order l gives the number of nodal lines on the surface

If the azimuthal order m is zero, the nodal lines are small circles about the pole. These are called zonal harmonics and do not depend on φ.

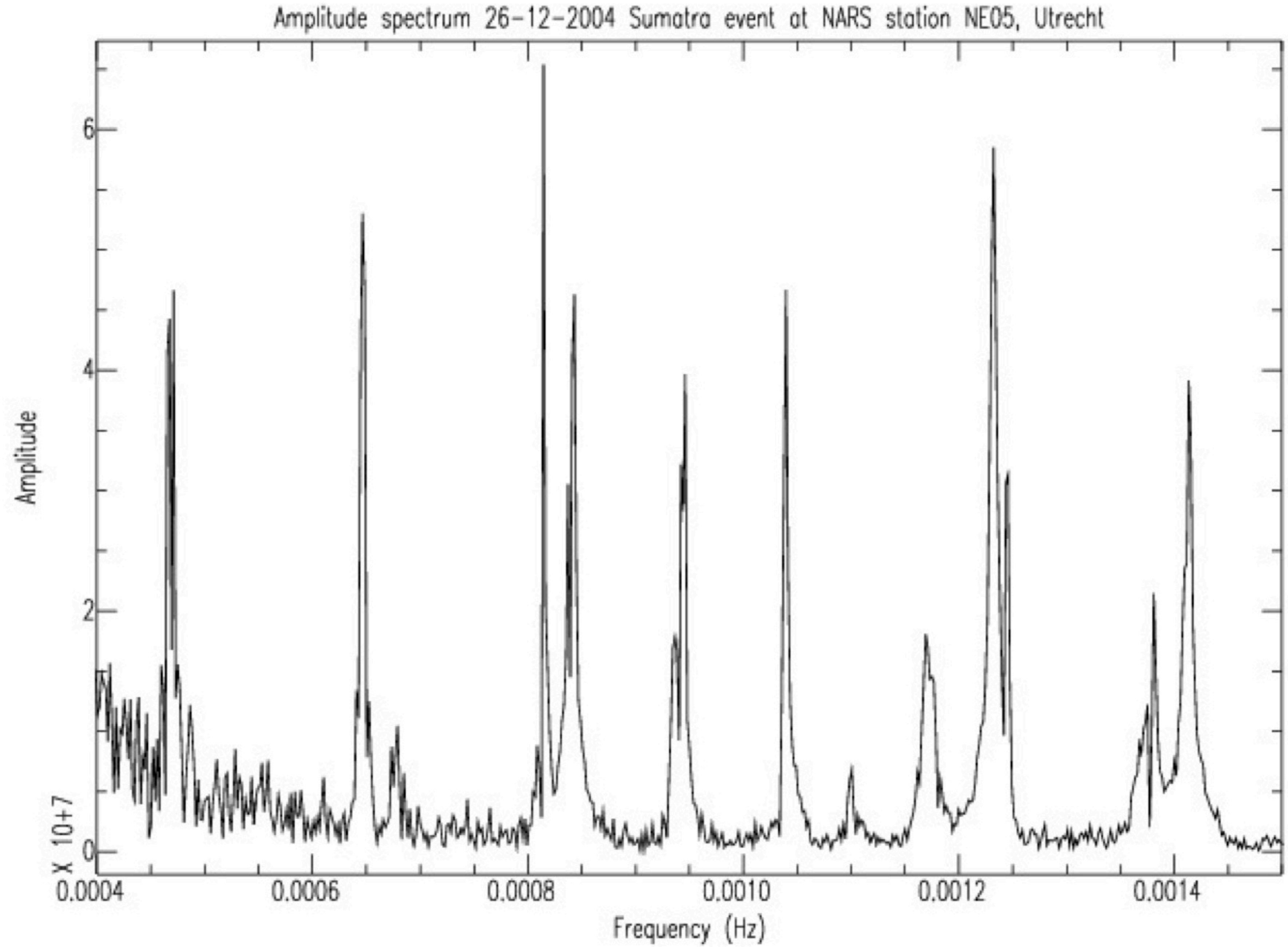
For a given angular order l, m has $2l+1$ values, leading to $2l+1$ different singlets (or eigenfunctions)

Spherical harmonics

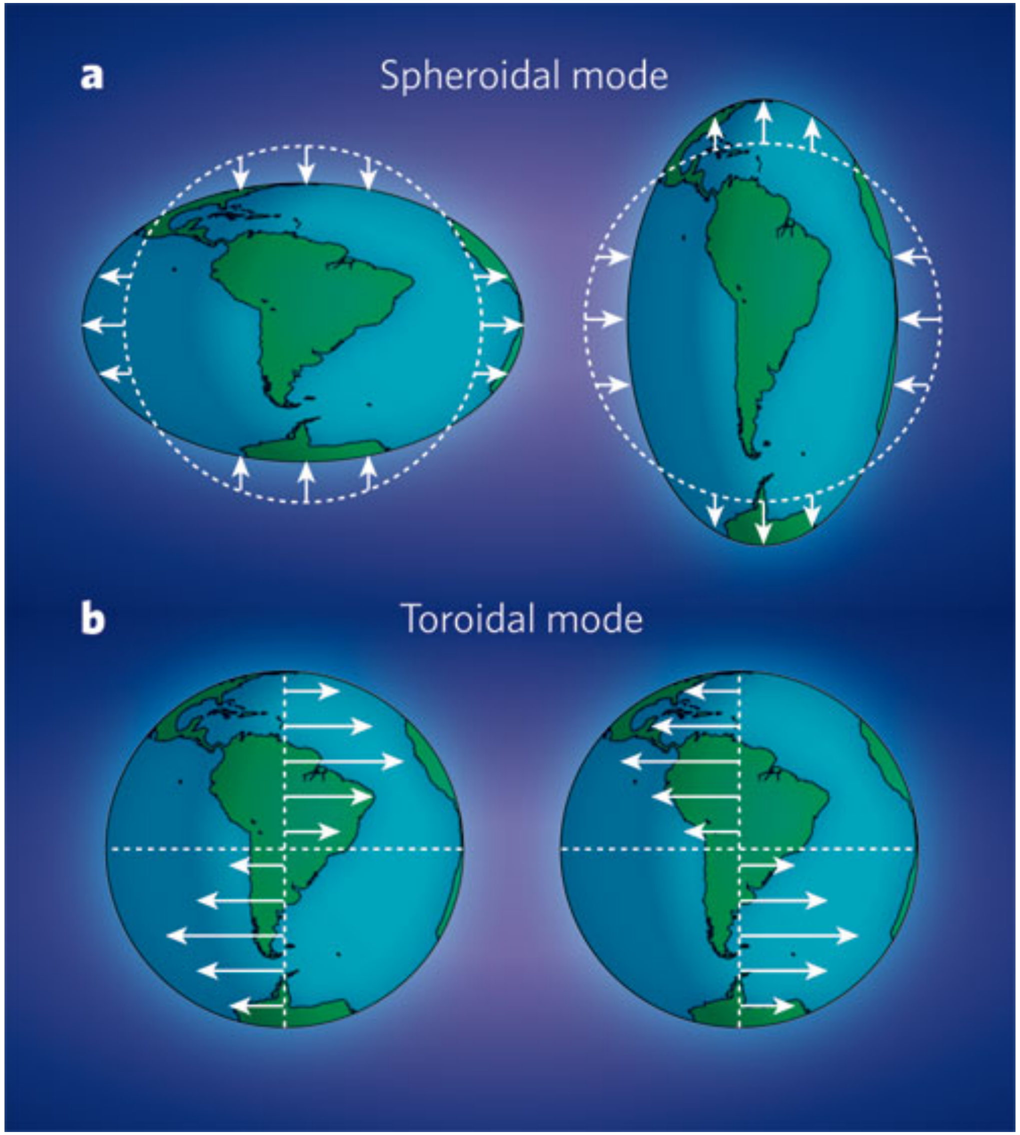


Sumatra earthquake 2004, M 9.1

Sumatra earthquake recorded in Utrecht

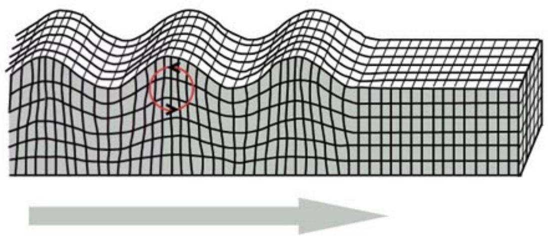


Displacement direction



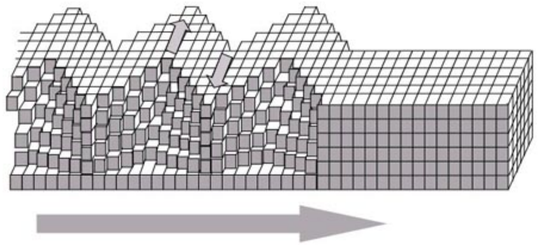
Spheroidal modes

- P-SV motion
- Similar to Rayleigh



Toroidal modes

- SH motion
- Similar to Love



Toroidal modes

For ${}_nT_l^m$:

n=radial order, l=angular order, m=azimuthal order

The $2l+1$ modes are different azimuthal orders $l=-m,-m+1,..,0,...,m$ are called singlets, and the group of singlets is called a multiplet.

If the Earth were perfectly spherically symmetric and non-rotating, all singlets in a multiplet would have the same frequency (called degeneracy).

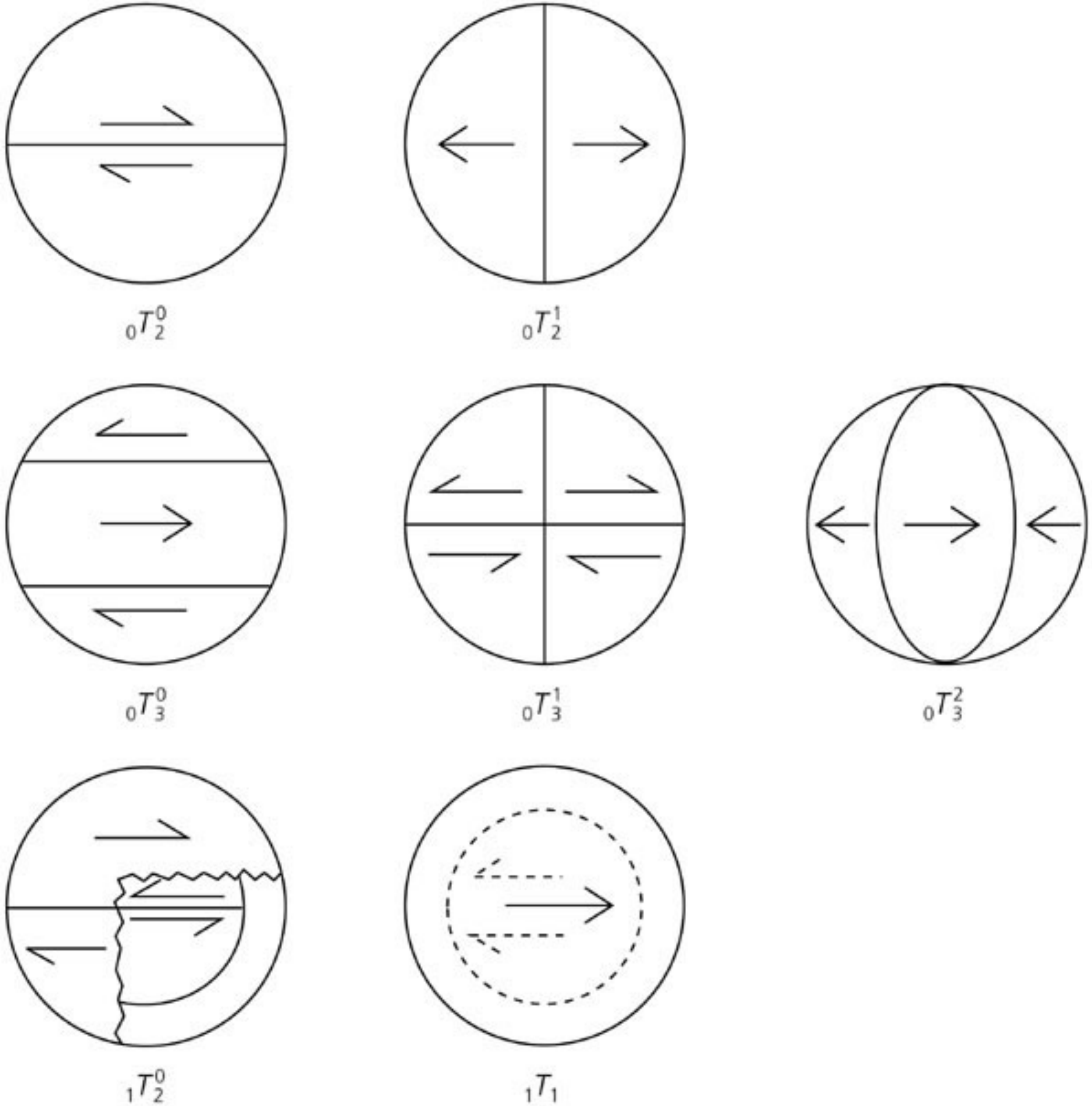
For example, the period of ${}_nT_l^0$ would be the same for ${}_nT_l^1$, ${}_nT_l^2$ etc.
In the real Earth, singlet frequencies vary (called splitting).

The splitting is usually small enough to ignore, so we drop the m superscript and refer to the entire ${}_nT_l^m$ multiplet at ${}_nT_l$ with eigenfrequency ${}_n\omega_l$

Toroidal modes

Toroidal modes with n=0 ($_0T_l$) are called fundamental modes (motions at depth in the same direction as at the surface).

Modes with n>0 are called overtones (motions reverse directions at different depths)



Spheroidal modes

For ${}_nS_l^m$:

n=radial order, l= angular order, m=azimuthal order

The 2l+1 modes are different azimuthal orders $l=-m,-m+1,..,0,\ldots,m$ are called singlets, and the group of singlets is called a multiplet.

If the Earth were perfectly spherically symmetric and non-rotating, all singlets in a multiplet would have the same frequency (called degeneracy).

For example, the period of ${}_nS_l^0$ would be the same for ${}_nS_l^1$, ${}_nS_l^2$ etc.
In the real Earth, singlet frequencies vary (called splitting).

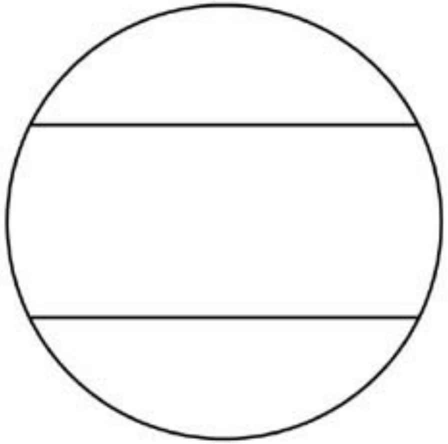
The splitting is usually small enough to ignore, so we drop the m superscript and refer to the entire ${}_nS_l^m$ multiplet at ${}_nS_l$ with eigenfrequency ${}_n\omega_l$.

Spheroidal modes

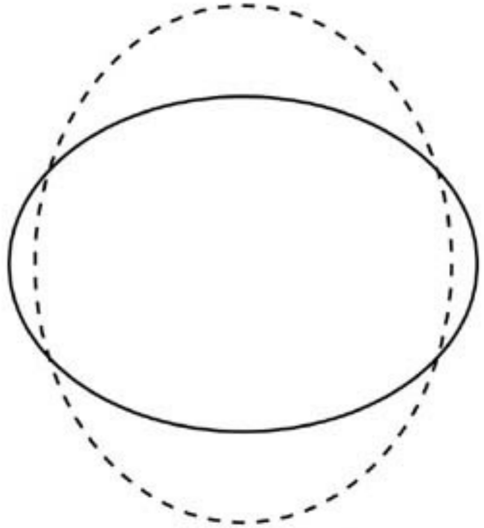
${}_0S_2$ (football mode) is the gravest (lowest frequency) mode, with a period of 3233 seconds, or 54 minutes.

There is no ${}_0S_1$ which would correspond to a lateral translation of the planet.

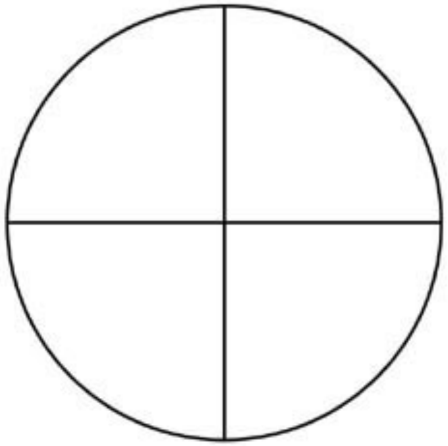
The ${}_1S_1$ Slichter mode due to lateral sloshing of the inner core through the liquid outer core, is not yet observed, but should have a frequency of about 5 ½ hours.



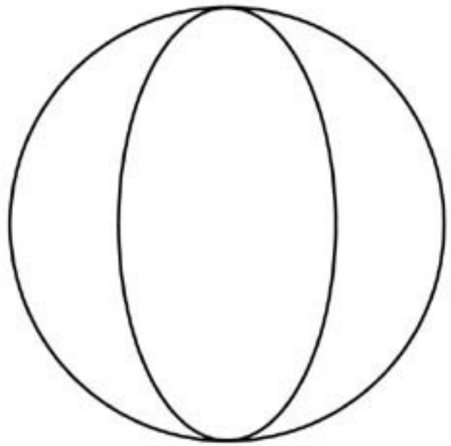
${}_0S_2^0$



${}_0S_2^0$ (motion)

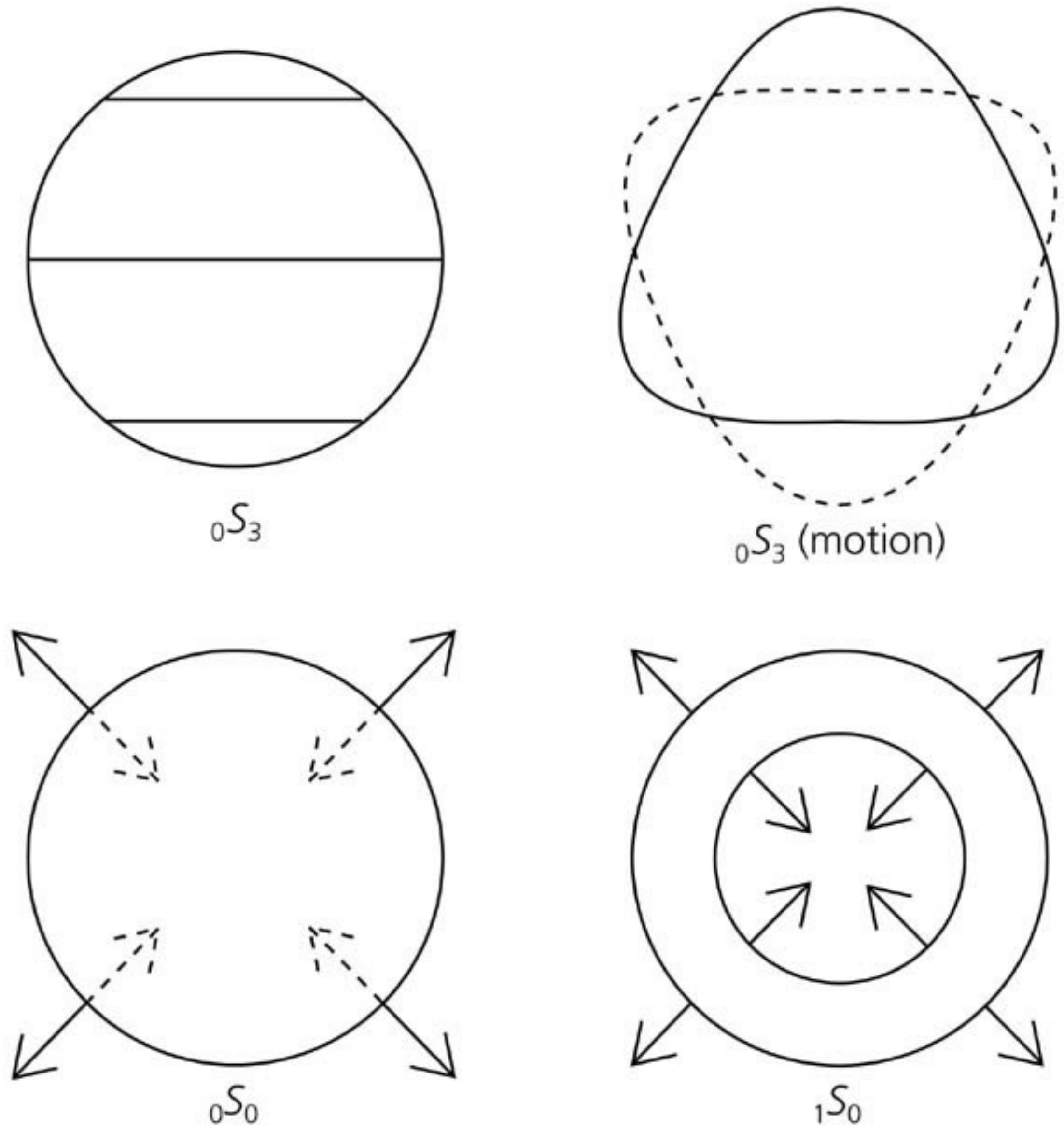


${}_0S_2^1$

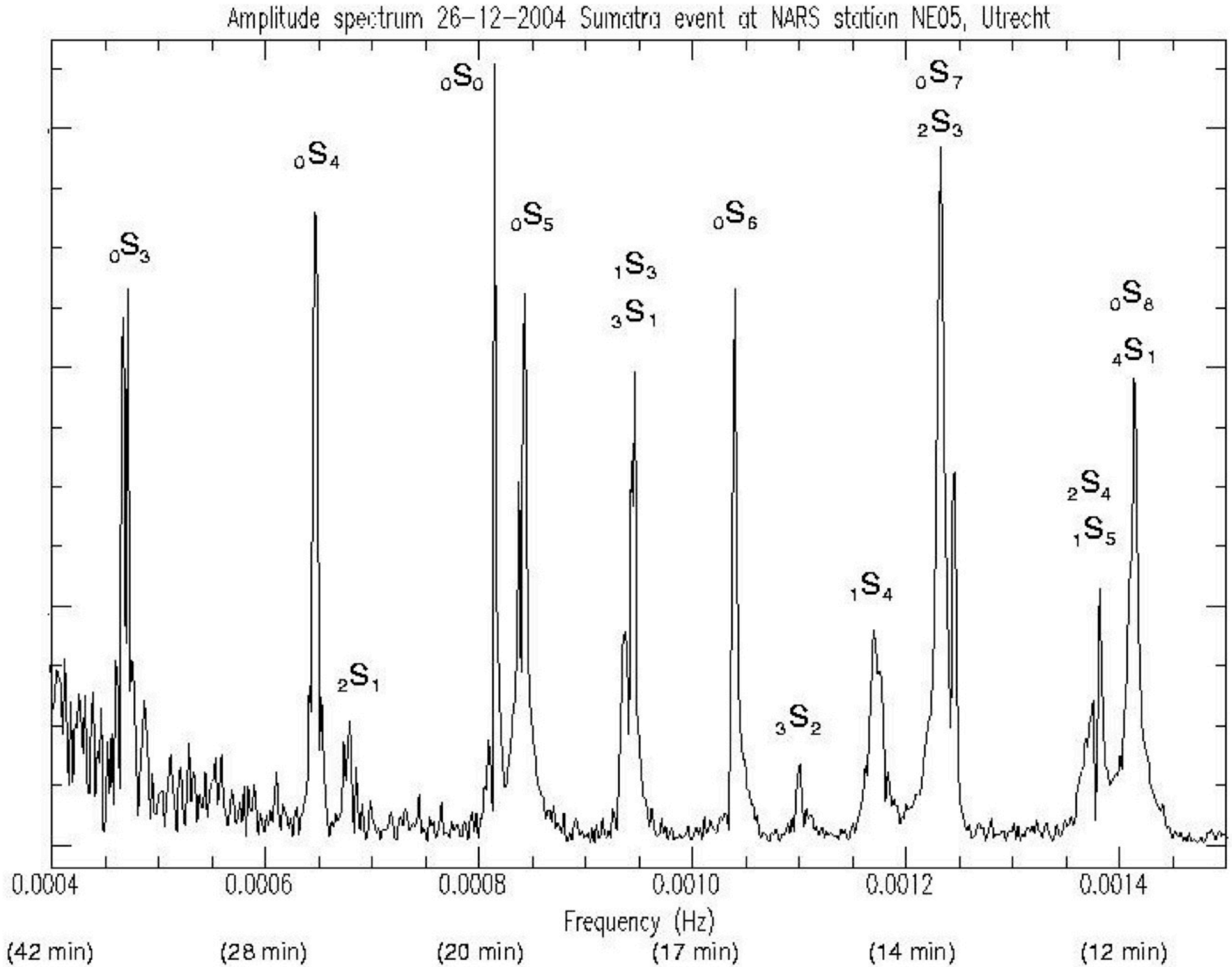


${}_0S_2^2$

Spheroidal modes



The “breathing” mode $_0S_0$ involves radial motions of the entire Earth that alternate between expansion and contraction.

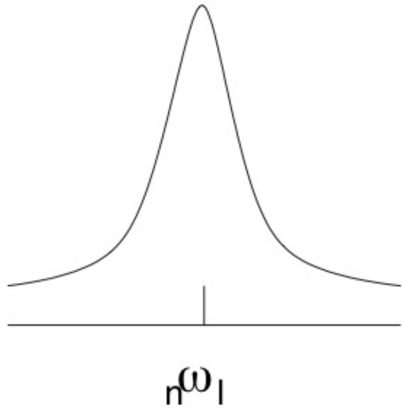


Eigenfrequencies of the earth excited by the Sumatra earthquake and recorded in Utrecht.

SNREI

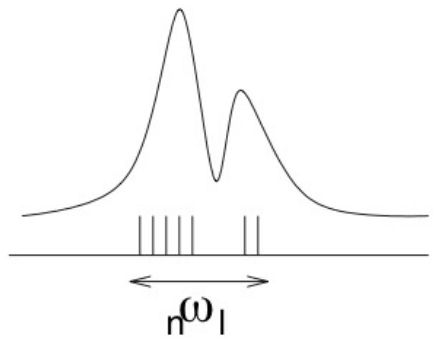
* spherical symmetric,
* non–rotating,
* isotropic Earth

degeneracy:
2l+1 singlets have same frequency



Real Earth

* Rotation
* Ellipticity
* Heterogeneity
* Anisotropy

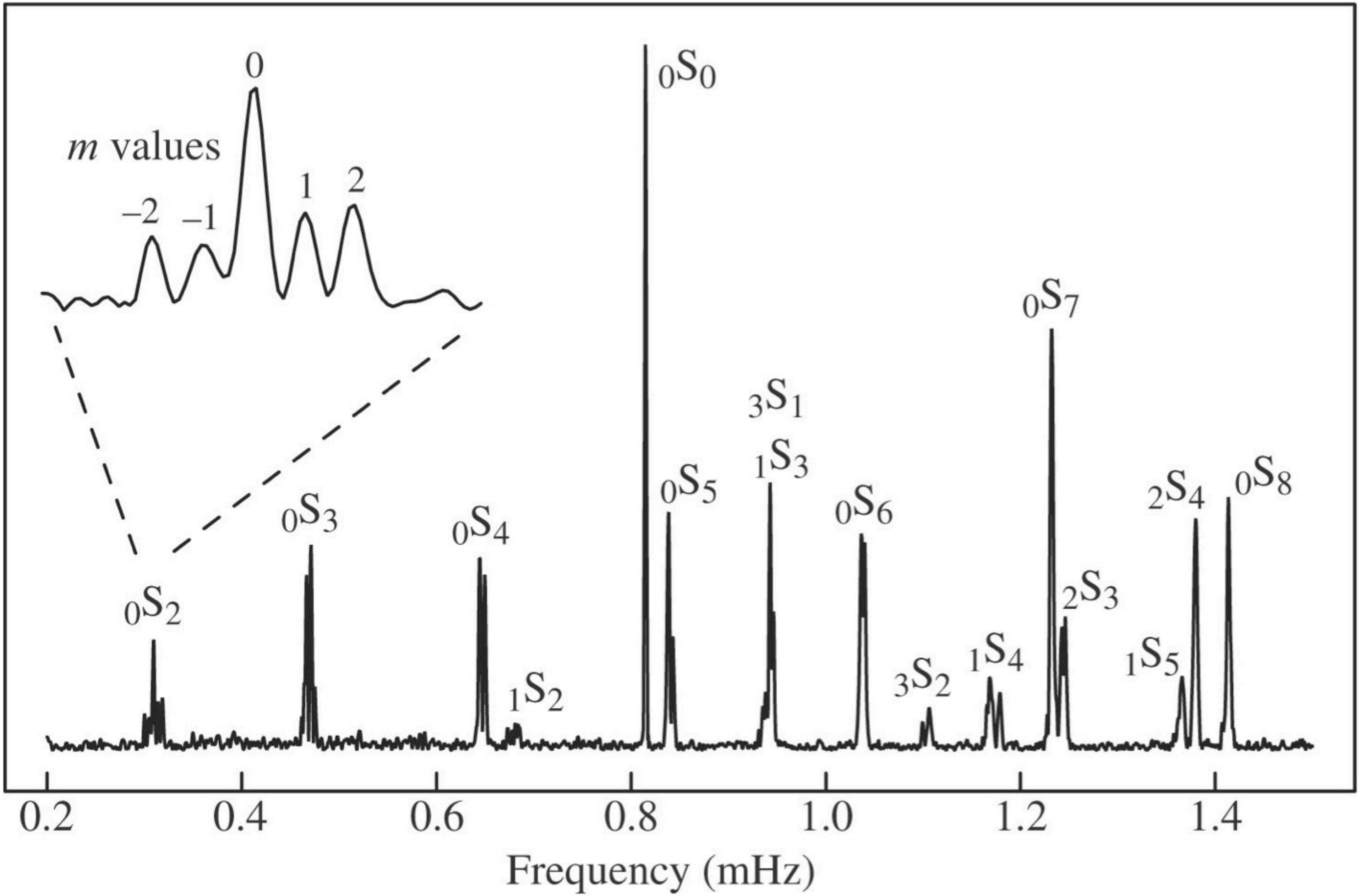


degeneracy removed:
2l+1 singlets have different frequency

↓

splitting and coupling

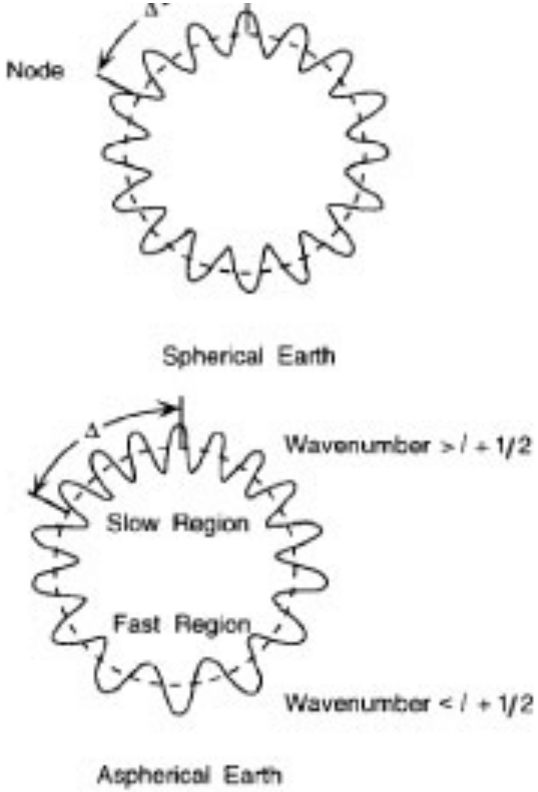
Singlet splitting

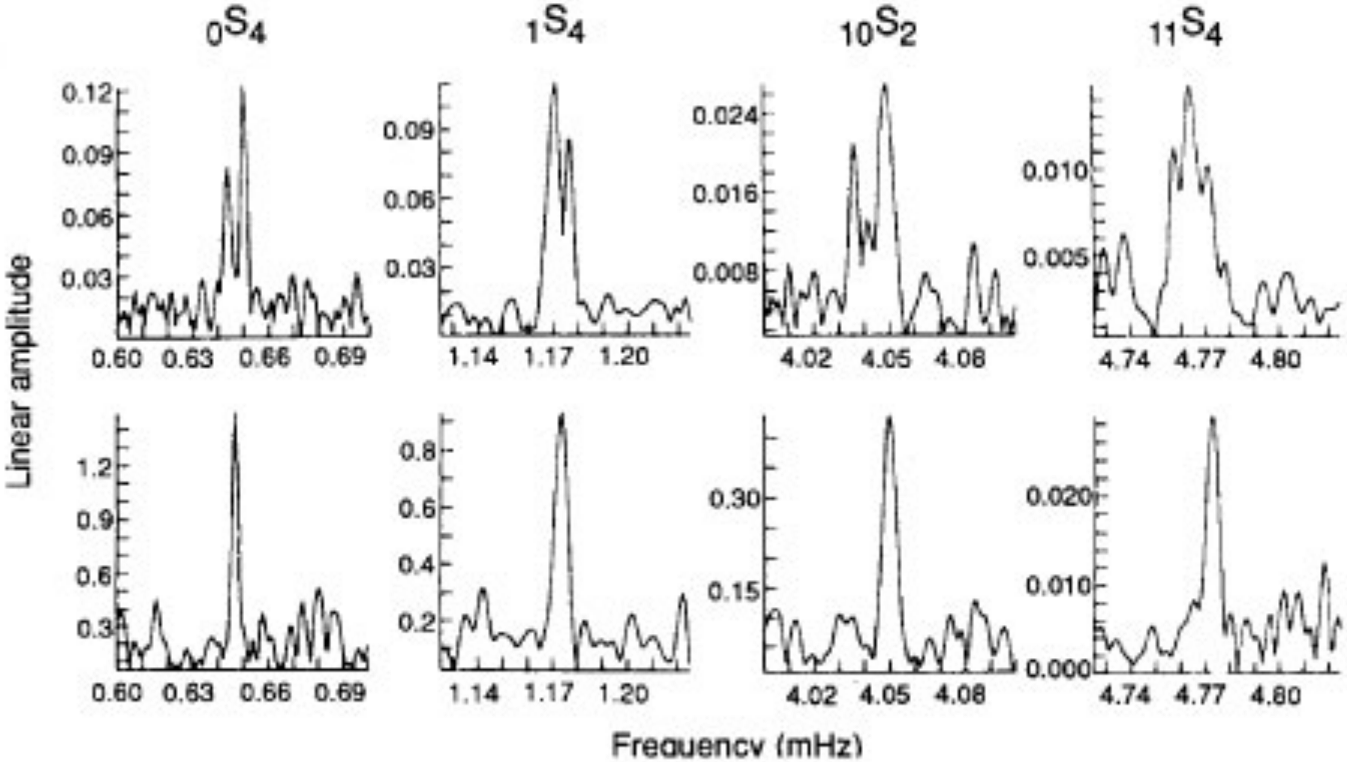


Splitting

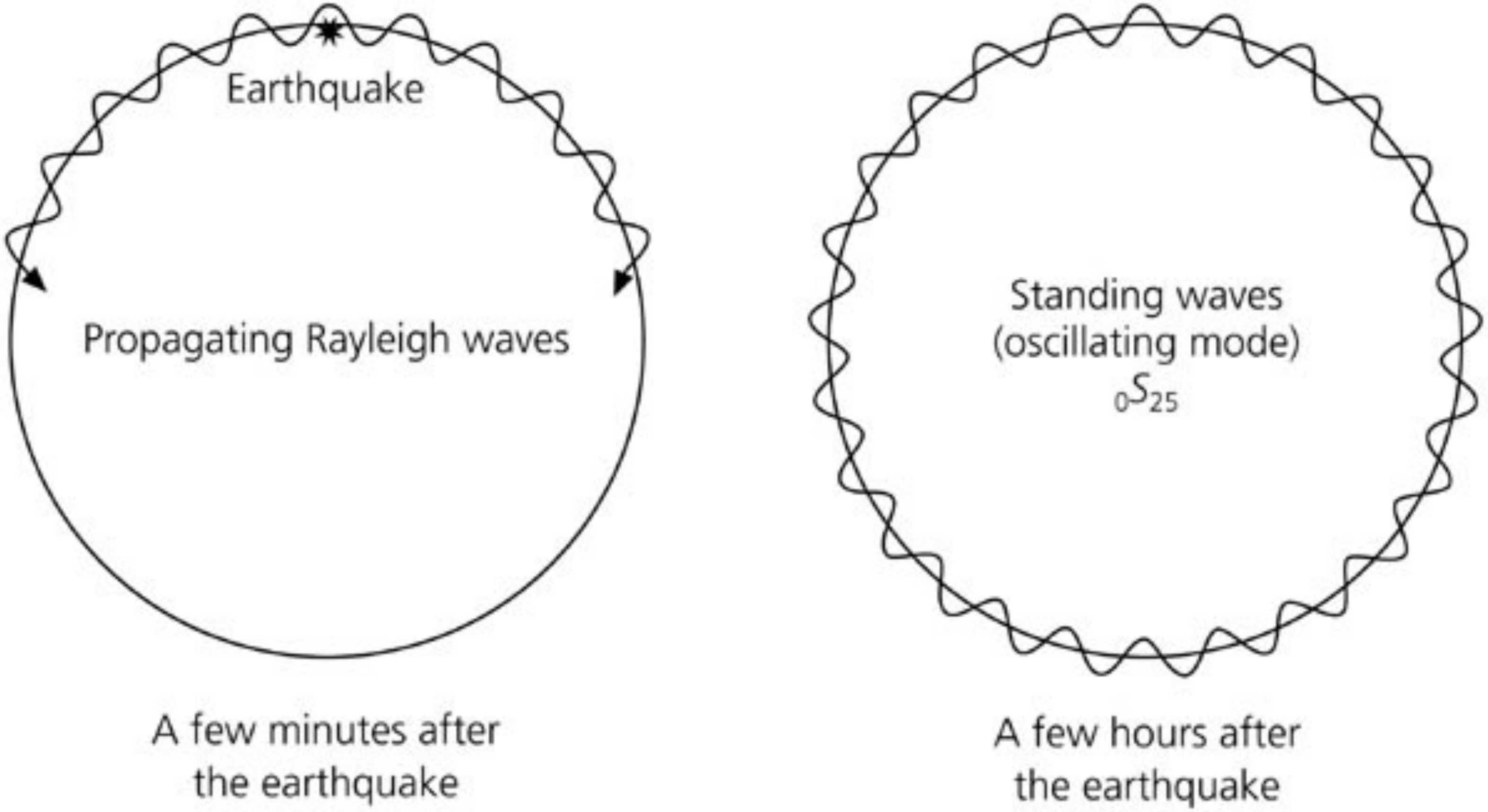
Split due to rotation

Anomalously split due to inner core anisotropy





Normal modes and surface waves



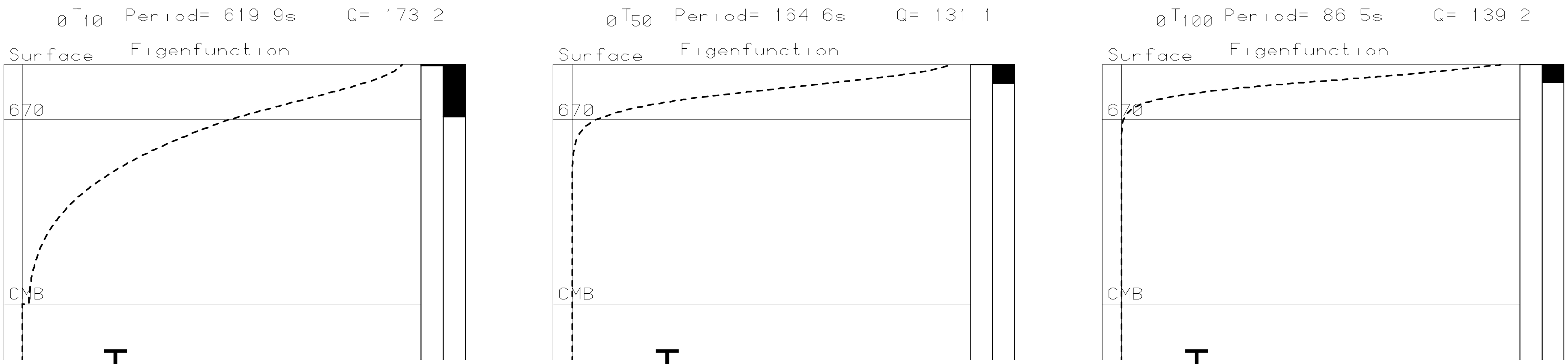
The mode with angular order l and frequency $_n\omega_l$ corresponds to a travelling wave with horizontal wavelength $\lambda_x = 2\pi / \mid k_x \mid = 2\pi a/(l + 1/2)$ that has l+1/2 wavelengths around the Earth.

These waves travel at horizontal phase velocity

$$c_x =_n \omega_l / \mid k_x \mid =_n \omega_l a/(l + 1/2)$$

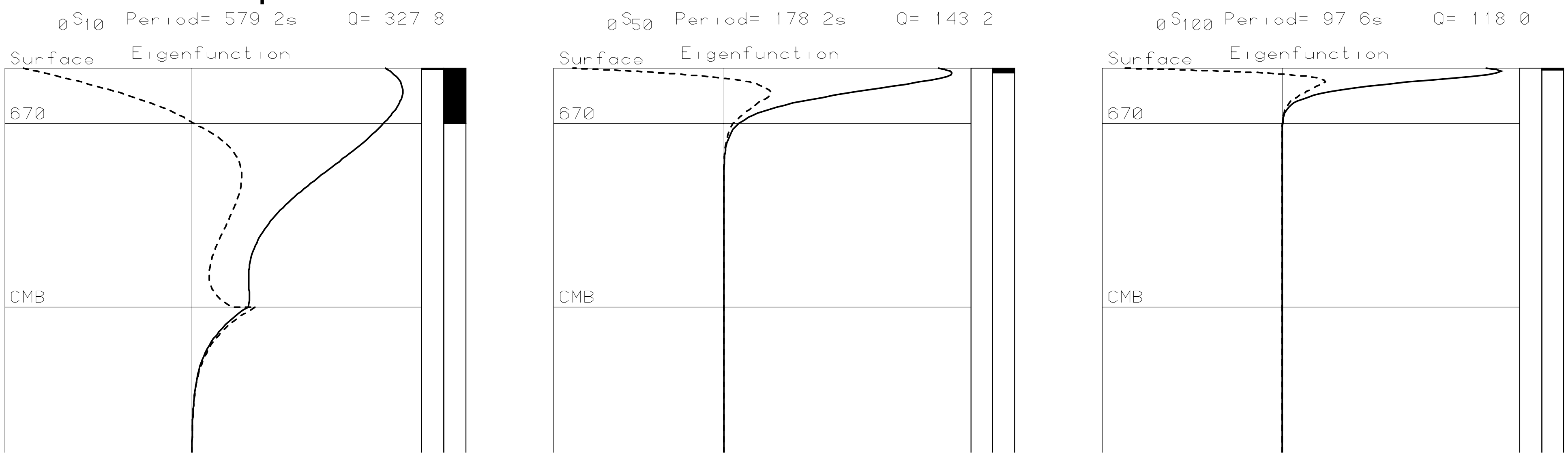
Eigenfunctions of fundamental modes

Fundamental toroidal branch



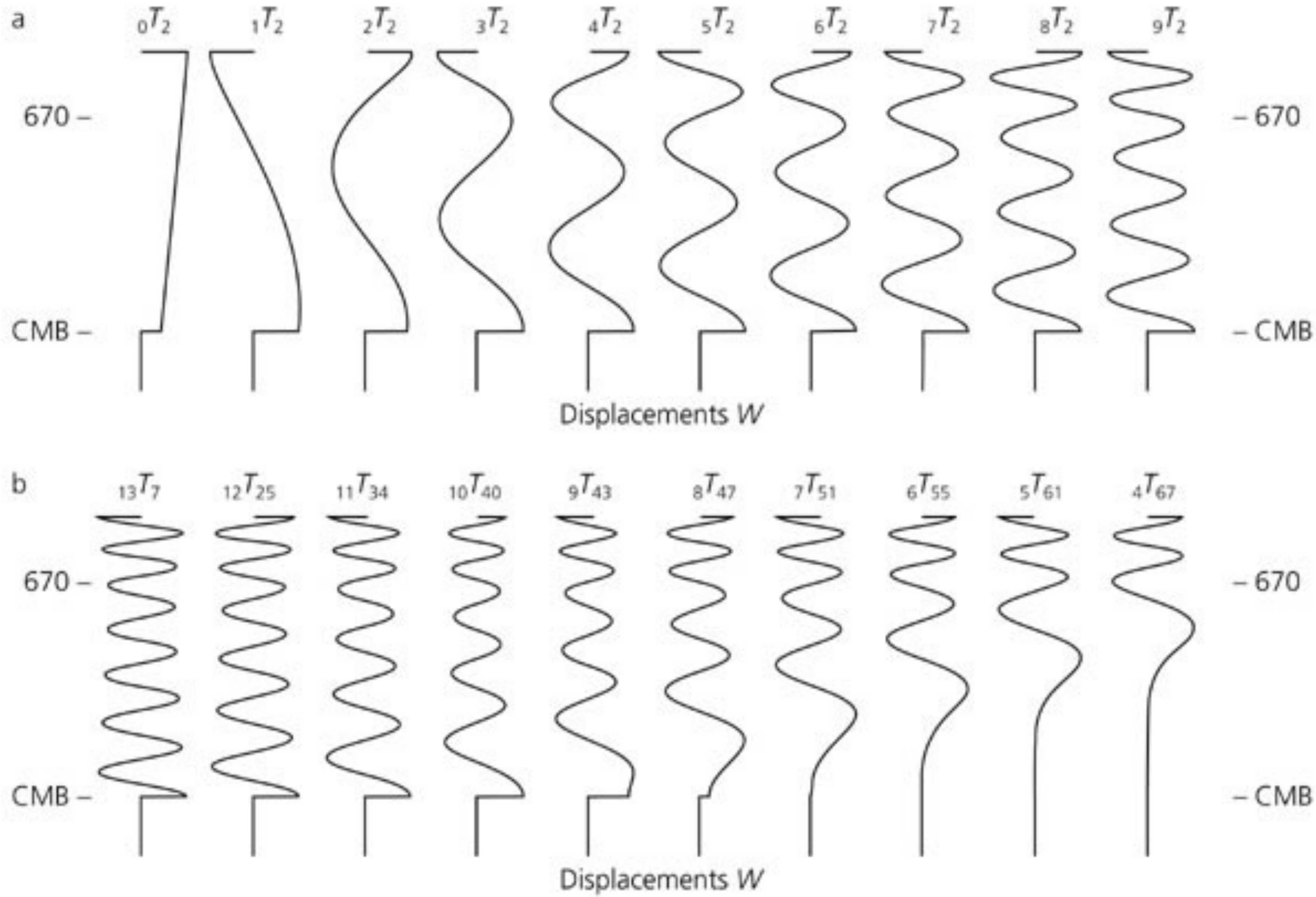
$_0T_{10}$ $_0T_{50}$ $_0T_{100}$

Fundamental spheroidal branch

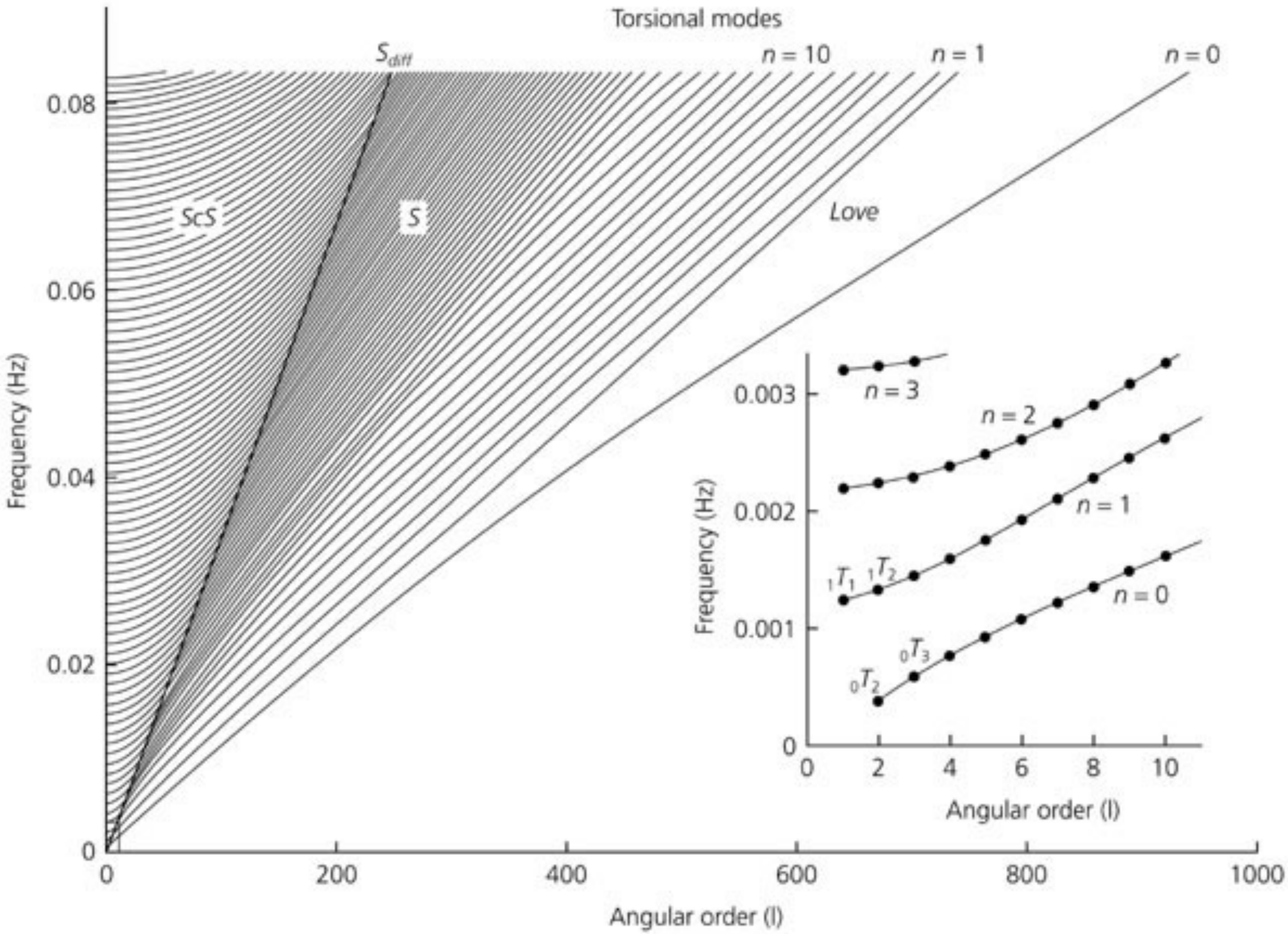


$_0S_{10}$ $_0S_{50}$ $_0S_{100}$

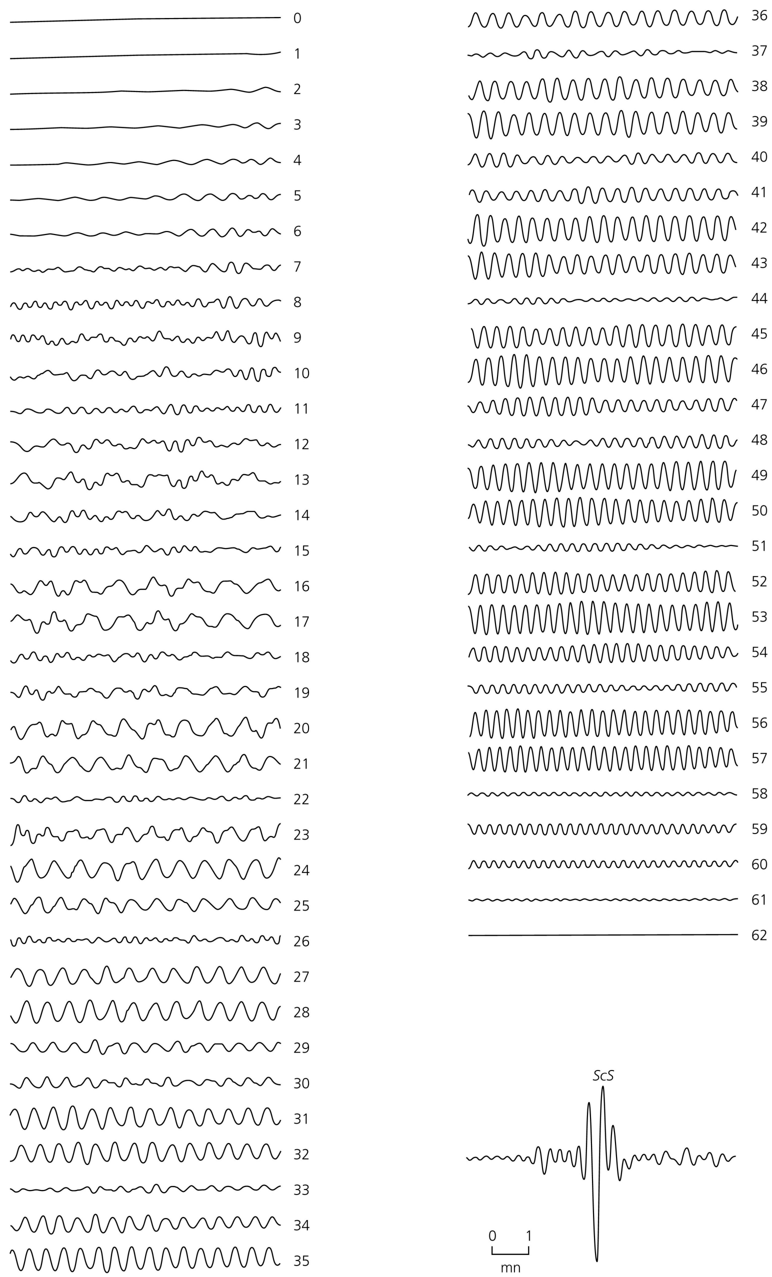
Eigenfunctions of overtones (higher modes)



Body waves by mode summation



ScS wave by mode summation



Application: density tomography

Only normal modes are sensitive to density perturbations in the Earth.