Exercises Introduction to seismology and seismics
1: Stress, strain, constitutive equation, seismic wave equation

1. For the stress tensor

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

   find the traction on
   (a) the $x - y$ plane
   (b) the $y - z$ plane
   (c) the plane with unit normal $\hat{\mathbf{n}} = (n_1, n_2, n_3)$.

2. For the stress tensor

$$\sigma = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   find the principal stresses and associated directions. (Note that the directions of principal stress should be perpendicular.)

3. Give an example of a strain tensor for which there is
   (a) an increase in volume
   (b) extension in the $x-$direction, but overall decrease in volume
   (c) shear strain but no volume change.

4. For the strain tensor

$$e = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

   find the corresponding stress tensor assuming an isotropic solid with Lamé constants λ and μ.

5. The University of California, San Diego, operates the Pinon Flat Observatory in the mountains northeast of San Diego. Instruments include high-quality strain meters for measuring crustal deformation.

   (a) Assume, at 5 km depth, the seismic velocities are $\alpha = 6.0$ km/s and $\beta = 3.5$ km/s and the density is $\rho = 2.7\ \mathrm{g/cm^3}$. Compute the values for the Lamé parameters, λ and μ, from these numbers. Express your answer in units of Pascals.

   (b) Following the 1992 Landers earthquake (magnitude 7.3), located in southern California 80 km north of the Observatory, the strain meters measured a large static change in strain compared to values before the event. Horizontal components of the strain tensor changed by the following amounts: $e_{11} = -0.26 \times 10^{-6}$, $e_{22} = 0.92 \times 10^{-6}$, $e_{12} = -0.69 \times 10^{-6}$. In this notation 1 is east, 2 is north, and extension is positive. You may assume that this strain change occured instantaneously at the time of the event. Assuming these strain values are also accurate at depth, use the result you obtained in part (a) to determine the change in stress due to the Landers earthquake at 5 km depth, that is, compute the change in σ_{11}, σ_{22}, and σ_{12}. Treat this as a two-dimensional problem assuming there is no strain in the vertical direction and no depth dependence of the strain.

(c) Compute the orientations of the principal strain axes (horizontal). Express your answers as azimuths (degrees east of north).

6. Derive the constitutive law (eq. 2.3.70 of the book) for an isotropic and linearly elastic material using the elastic tensor given by eq. 2.3.69. (In other words: explain how eq. 2.3.70 is obtained using eq. 2.3.69.)

7. Derive the ratio of P-to-S wave velocities in a Poisson solid.

8. Reproduce the derivation of the seismic wave equation (eq. 2.4.12) using the equation of motion, expressions for the stress and strain tensor, and the constitutive equation for a linearly elastic isotropic material.

Exercises Introduction to seismology and seismics, 2. Wave propagation

1. An alternative to using potentials to find seismic wave solutions to the equation of motion in terms of displacements is to formulate wave equations for the dilatation and curl of the displacement field. To see this:

   (a) Take the divergence of

   $$(\lambda + 2\mu)\nabla(\nabla \cdot \bar{u}) - \mu(\nabla \times (\nabla \times \bar{u})) = \rho \frac{\partial^2 \bar{u}}{\partial t^2}$$

   to obtain a wave equation for the dilatation θ. At what velocity does θ propagate?

   (b) Take the curl of the equation given in (a) to obtain a wave equation for $\nabla \times \bar{u}$. At what velocity does $\nabla \times \bar{u}$ propagate?

2. Consider two types of monochromatic plane waves propagating in the x-direction in an isotropic medium:
   (a) P-wave for which $u_x(\bar{x}, t) = A\sin(\omega t - kx)$,
   (b) S-wave with displacement in the y direction: $u_y(\bar{x}, t) = A\sin(\omega t - kx)$,
   For each case, derive expressions for the nonzero components of the stress tensor. To do this, first get the components of the strain tensor.

3. For waves propagating in an arbitrary direction given by the wavenumber $\bar{k}$,

   (a) Show that the P-wave displacement due to the scalar potential

   $$\phi(\bar{x}, t) = e^{i(\omega t - \bar{k}\cdot\bar{x})}$$

   is parallel to the propagation direction.

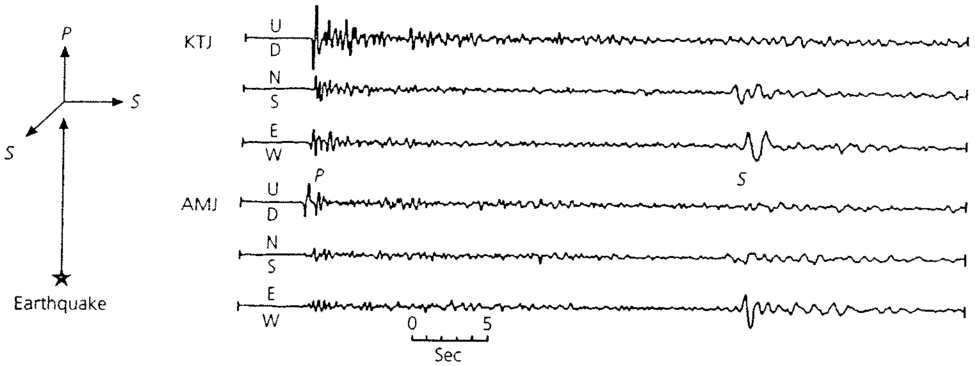
   (b) Show that the S-wave displacement due to the vector potential

   $$\bar{\Upsilon}(\bar{x}, t) = \bar{A}e^{i(\omega t - \bar{k}\cdot\bar{x})}$$

   is perpendicular to the propagation direction.

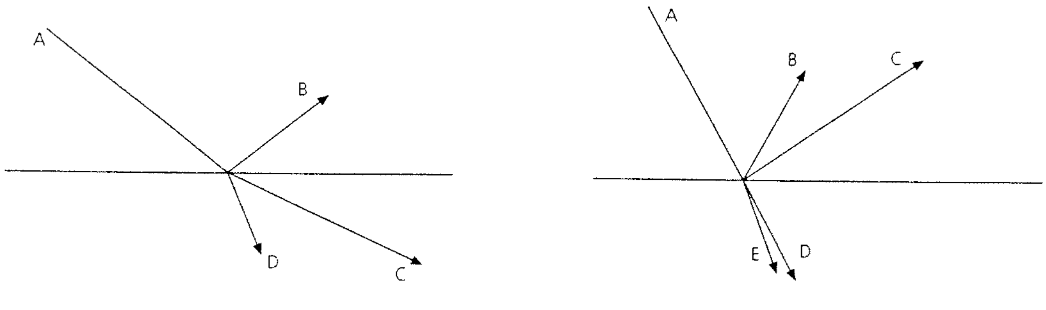
4. On a seismometer located at the surface very close to the earthquake location, the P and S wave traveling down and reflected back from the core, PcP and ScS, arrive at 8 minutes, 31 seconds, and 15 minutes, 36 seconds, respectively after the earthquake. If the earth's radius is 6371 km and the core's radius is 3480 km:
   (a) Find the average P- and S-wave velocities in the earth's mantle.
   (b) Use these average velocities to estimate if the mantle is close to a Poisson solid.

5. Estimate the P- and S-wave velocities in the upper part of the mantle by assuming it is a Poisson solid, and that the earthquake for which the seismograms are shown in the figure below occurred at a depth of 280 km. Compare these velocities to the average mantle values obtained in problem 4. (Note that the seismograms do not start at the origin time of the earthquake.)



Three-component seismograms at two stations from an earthquake beneath Japan. Because the stations are nearly above the earthquake, the *P* wave has its largest amplitude on the vertical (U–D, "Up"–"Down") components. (Ando *et al.*, 1983. *J. Geophys. Res.*, *88*, 5850–64, copyright by the American Geophysical Union.)

6. For the two cases of an incident wave hitting a plane boundary between to media shown in the figures below,
(a) Determine which waves are P waves and which are S waves.
(b) Determine which media are liquid (i.e. $\mu = 0$) and which are solid.
(c) For the two media in each case, determine which has the higher P-wave velocity.



7. For spherically symmetric or layered earth models, P-SV wave motion separates completely from SH motion. Despite this, P-waves are often weakly observed on the transverse (or "y-") component. Give several reasons why this might occur.

Exercises Introduction to seismology and seismics,
3a. Reflection and transmission

1. Calculate the normal incidence P-wave reflection coefficients for the following interfaces:
   (a) Sandstone ($\alpha = 3.0\ km/s,\ \rho = 2.2\ g/cm^3$) above limestone ($\alpha = 4.1\ km/s,\ \rho = 2.2\ g/cm^3$).
   (b) A possible crust-mantle interface of granulite ($\alpha = 7.3\ km/s$, $\rho = 3.2\ g/cm^3$) over peridotite ($\alpha = 8.1\ km/s,\ \rho = 3.3\ g/cm^3$).
   (c) An interface between water ($\alpha = 1.5\ km/s,\ \rho = 1.0\ g/cm^3$) and "hard" seafloor ($\alpha = 4.0\ km/s,\ \rho = 2.5\ g/cm^3$) in a marine seismic setting.
   (d) A P-wave incident from below at the free surface.

2. From "A guided tour of mathematical physics" by R. Snieder:

Reflection and transmission by a stack of layers

Lord Rayleigh[1] addressed in 1917 the question why some birds or insects have beautiful iridescent colors. He explained this by studying the reflective properties of a stack of thin reflective layers. This problem is also of interest in geophysics; in exploration seismology one is also interested in the reflection and transmission properties of stacks of reflective layers in the earth. Lord Rayleigh solved this problem in the following way. Suppose we have one stack of layers on the left with reflection coefficient R_L and transmission coefficient T_L and another stack of layers on the right with reflection coefficient R_R and

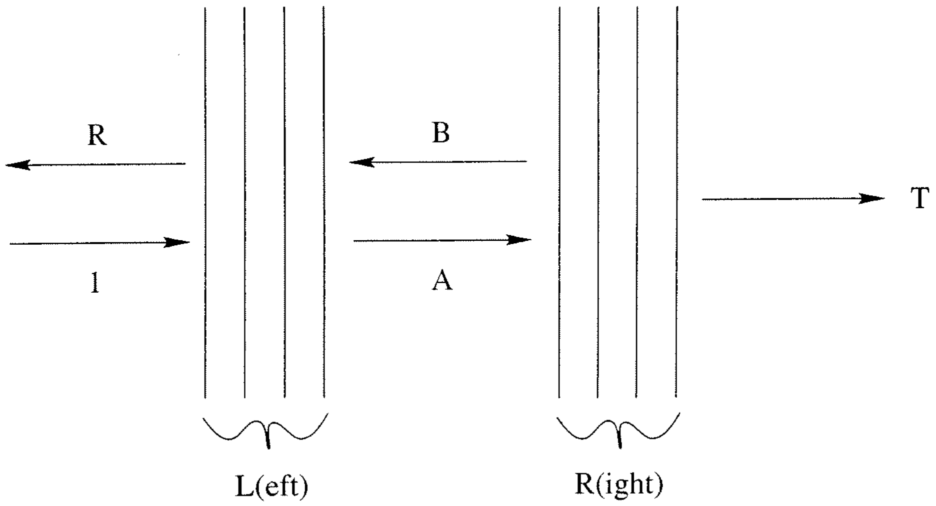


Figure 2.4: Geometry of the problem where stacks of n and m reflective layers are combined. The notation of the strength of left- and rightgoing waves is indicated.

[1] Rayleigh, Lord, 1917, On the reflection of light from a regularly stratified medium, *Proc. Roy. Soc. Lon.*, *A93*, 565-577.

transmission coefficient T_R. If we add these two stacks together to obtain a larger stack of layers, what are the reflection coefficient R and transmission coefficient T of the total stack of layers? See figure (2.4) for the scheme of this problem. Note that the reflection coefficient is defined as the ratio of the strength of the reflected wave and the incident wave, similarly the transmission coefficient is defined as the ratio of the strength of the transmitted wave and the incident wave. For simplicity we will simplify the analysis and ignore that the reflection coefficient for waves incident from the left and the right are in general not the same. However, this simplification does not change the essence of the coming arguments.

Before we start solving the problem, let us speculate what the transmission coefficient of the combined stack is. Since the transmission coefficient T_L of the left stack determines the ratio of the transmitted wave to the incident wave, and since T_R is the same quantity of the right stack, it seems natural to assume that the transmission coefficient of the combined stack is the product of the transmission coefficient of the individual stacks: $T = T_L T_R$. However, this result is wrong and we will try to discover why this is so.

Consider figure (2.4) again. The unknown quantities are R, T and the coefficients A and B for the right-going and left-going waves between the stacks. An incident wave with strength 1 impinges on the stack from the left. Let us first determine the coefficient A of the right-going waves between the stacks. The right-going wave between the stacks contains two contributions; the wave transmitted from the left (this contribution has a strength $1 \times T_L$) and the wave reflected towards the right due the incident left-going wave with strength B (this contribution has a strength $B \times R_L$). This implies that:

$$A = T_L + BR_L \ . \tag{2.25}$$

Problem a: Using similar arguments show that:

$$B = AR_R \ , \tag{2.26}$$

$$T = AT_R \ , \tag{2.27}$$

$$R = R_L + BT_L \ . \tag{2.28}$$

This is all we need to solve our problem. The system of equations (2.25)-(2.28) consists of four linear equations with four unknowns A, B, R and T. We could solve this system of equations by brute force, but some thinking will make life easier for us. Note that the last two equations immediately give T and R once A and B are known. The first two equations give A and B.

Problem b: Show that

$$A = \frac{T_L}{(1 - R_L R_R)} \ , \tag{2.29}$$

$$B = \frac{T_L R_R}{(1 - R_L R_R)} . \tag{2.30}$$

This is a puzzling result, the right-going wave A between the layers does not only contain the transmission coefficient of the left layer T_L but also and additional term $1/(1 - R_L R_R)$.

Problem c: Make a series expansion of $1/(1 - R_L R_R)$ in the quantity $R_L R_R$ and show that this term accounts for the waves that bounce back and forth between the two stacks. Hint: use that R_L gives the reflection coefficient for a wave that reflects from the left stack, R_R gives the reflection coefficient for one that reflects from the right stack so that $R_L R_R$ is the total reflection coefficient for a wave that bounces once between the left and the right stack.

This implies that the term $1/(1 - R_L R_R)$ accounts for the waves that bounce back and forth between the two stacks of layers. It is for this reason that we call this term a *reverberation* term. It plays an important role in computing the response of layered media.

Problem d: Show that the reflection and transmission coefficient of the combined stack of layers is given by:

$$R = R_L + \frac{T_L^2 R_R}{(1 - R_L R_R)}, \tag{2.31}$$

$$T = \frac{T_L T_R}{(1 - R_L R_R)}. \tag{2.32}$$

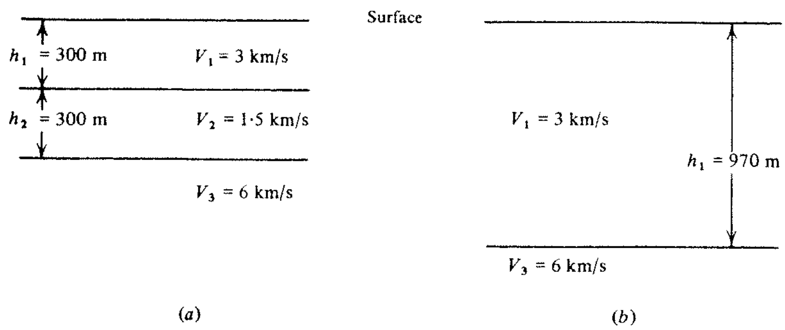
In the beginning of this section we conjectured that the transmission coefficient of the combined stacks is the product of the transmission coefficient of the separate stacks.

Problem e: Is this correct? Under which conditions is it approximately correct?

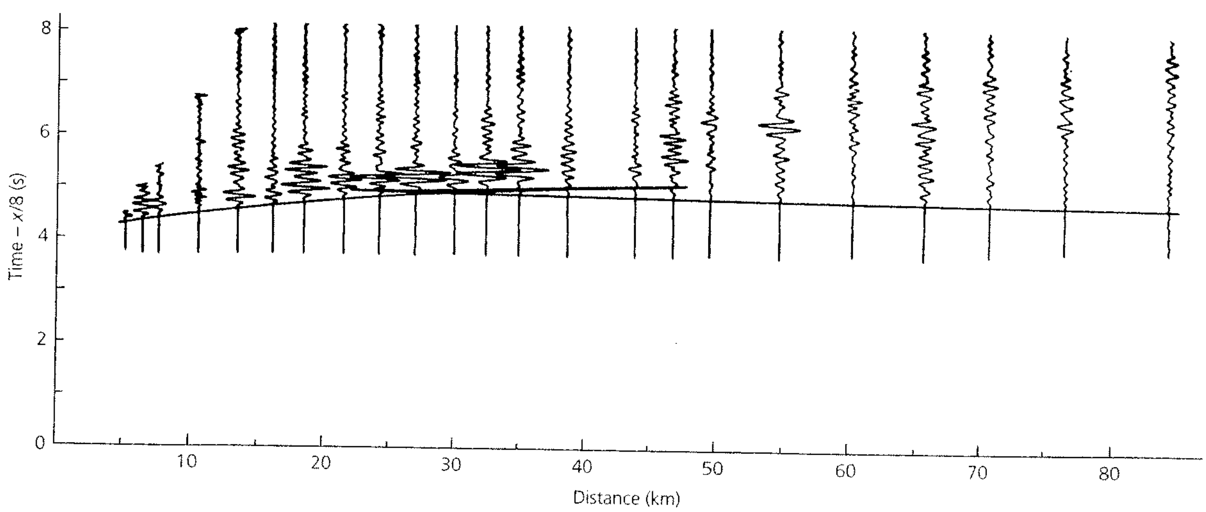
Problem f: Given your results for exercise 1, explain which reverberations between two interfaces will have the largest amplitudes.

Exercises Introduction to seismology and seismics,
3b. Refraction seismology

1. (a) For a case of two layers overlying a halfspace, derive the expression for the travel time of the deepest head wave.



   (b) Show that the two geological sections illustrated in the above figure produce the same time-distance curve. Explain how this can be.



2. Analyze the data of the marine refraction experiment shown in the figure above, assuming for simplicity that the structure consists of a water layer, a crustal layer, and a mantle halfspace.
   (a) Assuming that the first arrivals are described by two line segments, for head waves at the top of the crust and mantle, find the corresponding velocities.
   (b) Although the direct wave traveling in the water layer is not shown, the P velocity for water is 1.5 km/s. Use the time intercept for the crustal head wave to find the water depth.
   (c) Use the time intercept for the P_n wave to find the crustal thickness.

3. To show that the head wave is predicted by Fermat's principle, consider a layer of thickness h with velocity v_0, overlying a halfspace with a higher velocity v_1.
   (a) Derive the travel time to the distance x for a head wave that is incident on the boundary at a (horizontal) distance y from the source, travels for some distance just below the boundary, and then returns to the surface at the same incidence angle at which it went down.
   (b) Find the y value giving the extremal travel time, and show that it corresponds to the critical angle of incidence.
   (c) Determine if this travel time is a minimum or a maximum.