Self-similar properties of avalanche statistics in a simple turbulent model

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In this paper, we consider a simplified model of turbulence for large Reynolds numbers driven by a constant power energy input on large scales. In the statistical stationary regime, the behaviour of the kinetic energy is characterized by two well-defined phases: a laminar phase where the kinetic energy grows linearly for a (random) time t_w followed by abrupt avalanche-like energy drops of sizes S due to strong intermittent fluctuations of energy dissipation. We study the probability distribution $P[t_w]$ and $P[S]$ which both exhibit a quite well-defined scaling behaviour. Although t_w and S are not statistically correlated, we suggest and numerically checked that their scaling properties are related based on a simple, but non-trivial, scaling argument. We propose that the same approach can be used for other systems showing avalanche-like behaviour such as amorphous solids and seismic events.

This article is part of the theme issue 'Scaling the turbulence edifice (part 1)'.

1. Introduction

It is well known that homogeneous and isotropic turbulence is characterized by strong intermittent bursts of energy dissipation. An extensive literature exists on the subject and we recommend [1] for an excellent introduction. Turbulence is not the only physical system

where energy dissipation occurs intermittently. Amorphous materials, for example, subject to constant shear rate, show avalanche-like events, which dissipate energy intermittently [2–4]. Analysis of their intermittency mostly focused on the statistical properties of the avalanche size S which usually shows a probability distribution $P[S] \sim S^{-\gamma}$ where $\gamma \in [1.25 : 1.5]$ depending on the material and its physical properties. Similarly, earthquake dynamics have been discussed in terms of earthquake magnitude or more properly in terms of the seismic moment M, which exhibits the celebrated Gutenberg–Richter Law $P[M] \sim M^{-(1+2b/3)}$, with $b \sim 1$ [5]. Another interesting quantity to consider is the interevent time t_w between avalanches [6,7], which has been poorly investigated in the past. Obviously, both S and t_w depend on the definition of the avalanche, a crucial point we will consider below.

The concept of avalanche or avalanche-like dynamics is not usually taken into account in turbulence for many reasons: energy dissipation is intermittent both in space and time; the statistical properties of energy dissipation are usually related to the intermittent fluctuations of energy transfer within the inertial range of turbulence; an avalanche does not seem appropriate to discuss energy transfer in the inertial range or in the dissipation range and so on. Thus, discussing turbulence in the framework of avalanche dynamics seems at least a rather exotic if not useless approach. However, we suggest that in some cases the dynamics of turbulent events may be investigated in terms of avalanche dynamics. In this paper, we provide an example based on a very simplified model of turbulence, namely a shell model. For this very special model, we show that avalanche dynamics can be identified and we are able to provide a well-defined meaning to the avalanche size S. Not surprisingly, S is related to the rate of energy dissipation. Also, we provide evidence that, for our case, both t_w and S should be considered *bulk quantities*: they are related to the forcing mechanism and do not depend on the statistical properties characterizing inertial range fluctuations. We explore similarities and differences in the statistics of avalanche dynamics in our model compared to results mostly observed in amorphous materials and for earthquakes. Among the similarities, we observe in our shell model a scaling behaviour of both S and t_w although with different scaling exponents compared to amorphous materials and earthquakes. More importantly, we observe a rather unusual scale invariance in our system which is also observed in other cases. To be precise, we look at the probability distribution $P[t_w|S_{\text{th}}]$ of the interevent times t_w for avalanches bigger than some threshold S_{th}. Upon increasing S_{th}, $P[t_w|S_{\text{th}}]$ remains invariant although t_w explicitly depends on S_{th}. This is also observed for amorphous materials [7–9] and it was first pointed out by Corral [6] by analysing earthquake catalogues (see [10] for a review). We also show strong evidence that, if scale invariance holds, then the scaling properties of S and t_w should be related, although both quantities are statistically independent in our model as well in the case of amorphous materials and earthquakes. These relations, obtained here for the first time, apply to our system and amorphous materials despite the difference in the scaling exponents. A speculative conjecture can then be made for earthquake dynamics with excellent agreement with the Corral results. Let us finally highlight that the concept of avalanche is quite general and that different approaches (such as the one proposed in [2,11]) are worthwhile exploring for various physical phenomena depending on the statistical properties of the system under consideration.

From the point of view of turbulence or turbulent flows, our approach may be generalized to describe other intermittent behaviours shown by *bulk quantities* whose physical description may be improved by exploiting the same approach outlined in this paper. As an example, we refer to the instability of a thermally driven system in a vertically elongated convection cell. Such a system can be mathematically modelled in terms of a fully periodic thermally driven Rayleigh–Benard cell where the presence of the so-called elevator modes leads to the growth of the kinetic energy associated with vertical motions. This exponential growth is then followed by sudden dissipation events mediated by a shear flow instability that redistributes such energy horizontally [12]. Another example can be found in the study of solar flares which, sometimes, are statistically investigated using models derived in the framework of self-organized criticality [11], i.e. in the framework of avalanche dynamics, see also [13] for a different point of view.

Our paper is organized as follows: in §2, we introduce our model and define the forcing mechanism. In §3, we discuss our definition of event or avalanche size S. In the §3(a),(b), we show that using a different definition of S based on level-crossing, no scale invariance is observed and illustrate the relevance of the forcing mechanism, respectively. All the numerical results are discussed in §3. In §4, we provide a theoretical analysis of our system and we show that if scale invariance holds, then there must be a relation between the scaling properties of t_w and S. We also discuss how our approach can be generalized to amorphous materials and earthquake dynamics with excellent agreement with experimental and/or numerical results. Some general conclusion is provided in §5. We further want to emphasize that our investigation provides a different point of view on turbulent flows complementary to the well-known properties of inertial range dynamics. We suggest that it represents a preliminary step in a new direction which is interesting to study.

2. Model equations

We consider a shell model [14] to describe our turbulent system which is defined by the equations:

$$\partial_t u_n = i(k_n u_{n+2} u_{n+1}^* - \delta k_{n-1} u_{n+1} u_{n-1}^* + (1-\delta)k_{n-2}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n, \qquad (2.1)$$

where u_n are complex variables, $k_n = 2^n$ and $\delta = 0.4$. This choice of δ is known to reproduce the scaling behaviour of $\langle |u_n|^p \rangle \sim k_n^{-\zeta(p)}$ with anomalous exponents $\zeta(p)$ in close agreement with the ones observed in three-dimensional turbulence. Thus the statistical properties of the inertial range in the models may be considered close to realistic. The crucial point in our case is the forcing term. We apply f_n to $n = 1, 2$ (i.e. large-scale forcing) with

$$f_n = \frac{A}{u_n^*} \quad n = 1, 2. \qquad (2.2)$$

Using (2.2) the rate of energy input ϕ in the system is simply given by $4A$ and it is constant. The equation for the kinetic energy $E(t) = \Sigma_n |u_n|^2$ takes the form:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \phi - \epsilon(t), \qquad (2.3)$$

where $\epsilon(t) = \nu \Sigma_n k_n^2 |u_n|^2$ is the rate of energy dissipation. For the numerical simulations discussed hereafter we chose $\nu = 10^{-9}$ and $A = 10^{-3}$. We emphasize that the following results are independent of the choice of ν and A. By injecting a constant power into our system, the signal for kinetic energy (figure 1b) shows two clearly different dynamical regimes, one in which the energy grows linearly in time (referred to as 'laminar' phase) and one in which abrupt energy losses occur (referred to as 'turbulent' phase). Here laminar and turbulent are used in a naive way just to distinguish between the two different regimes. One can formally use equation (2.3) to highlight an analogy between our model and similar studies done in amorphous solids. In the latter case, the system is driven by a constant shear rate $\dot{\Gamma}$. In amorphous solids, driven by a constant shear rate $\dot{\Gamma}$, the energy grows as $\Gamma \Sigma$, where Γ is the strain, whereas Σ is the internal stress. It follows that energy grows quadratically in time. One can use the Maxwell model to relate Σ and $\dot{\Gamma}$ usually written as

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = G_0(\dot{\Gamma} - f\Sigma), \qquad (2.4)$$

where G_0 is the elastic modulus and f is referred to as 'fluidity' (dimension $1/\text{time}$). The fluidity f can also be interpreted as the rate of plastic events in the system. Thus during the elastic branch (where the fluidity can be neglected) the stress Σ grows linearly in time. Obviously, the equation for the fluidity f should be provided to complete the problem description. In turbulence, at least within our simplified model, there is no 'global' variable such as stress Σ and there exists no equation (2.4). An analogy with equation (2.4) can nevertheless be found using the total energy E as a global quantity. Writing $\epsilon(t) = FE$ the similarity between (2.4) and (2.3) is more apparent. Here the definition of as $F = \epsilon/E$ does not provide any new physics and it is just a way to make a simple analogy. This analogy is just formal because the two systems (turbulence and amorphous
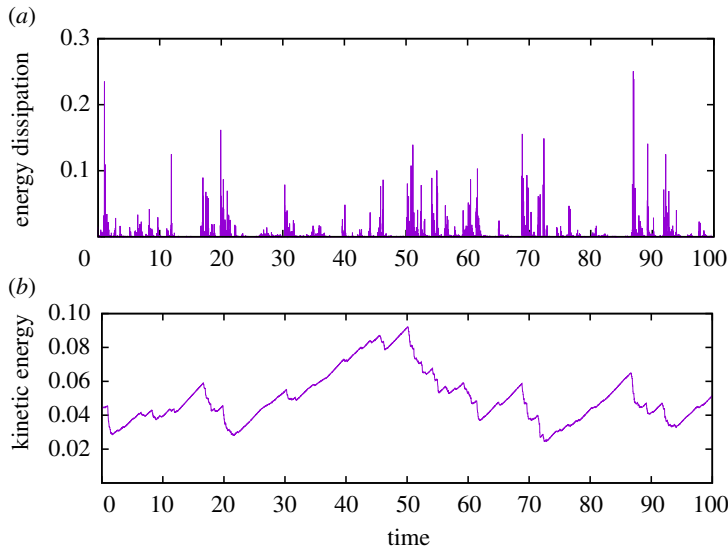
Figure 1. (*a*) Rate of energy dissipation $\epsilon(t)$ as a function of t corresponding to equation (2.1) at constant power input. (*b*) Behaviour of the kinetic energy $E(t)$ for the same time window. Looking at $E(t)$, we can easily distinguish two different dynamical phases in the system: a laminar phase where $E(t) \sim t$ and an avalanche phase where we observe drops of $E(t)$ corresponding to the bursts in the energy dissipation. (Online version in colour.)

solid) obey completely different physics and the variable F is not at all related to any plastic flow. In fact, as we shall see, the scaling observed for the avalanche sizes and their interevent time are completely different with respect to the one observed in amorphous solid.

In figure 1*a*, we also report the signal corresponding to the energy dissipation for the same time window, which shows, as expected, a strongly intermittent behaviour. Notice the link between the strong fluctuations occurring in the upper signal and the kinetic energy loss in the lower signal.

3. Event definition and scale invariance

Having in mind the signal shown in figure 1, we define an avalanche as the event for which $dE/dt < 0$. The starting point of an event is identified by a change in the sign of the derivative of kinetic energy in time, from positive to negative; meanwhile, the opposite change in sign of dE/dt identifies the end of the avalanche or energy drop. In figure 2, we illustrate our definition. For each event we compute the avalanche size as $S(k) = -\int_{t_i(k)}^{t_f(k)} dt(dE/dt)$, where k is a label for the event ($k = 1, 2, \ldots$), $t_i(k)$ is the initial time of the k event and $t_f(k)$ its corresponding final time. For $t \in [t_i(k), t_f(k)]$, we require that $dE/dt < 0$. Thus $t_i(k)$ is identified by the condition $\epsilon(t_i(k)) > \phi$ and $t_f(k)$ by the condition $\epsilon(t_f(k)) \leq \phi$. $S(k)$ thus represents the total energy release during avalanche event k.

Having thus defined avalanches, we can look at the probability distributions of the avalanche size S and the interevent times t_w The latter is defined as $t_w(k) = t_i(k) - t_f(k-1)$. In figure 3, we show $P[S]$ (*b*) and $P[t_w]$ (*a*): both probability distributions show very clear scaling regions over several decades. We indicate the corresponding scaling exponents by γ and α:

$$P[t_w] \sim \frac{1}{t_w^\alpha} \quad \text{and} \quad P[S] \sim \frac{1}{S^\gamma}. \tag{3.1}$$

A best fit estimate gives $\alpha = 1.65$ and $\gamma = 0.35$ with an accuracy of the order of a few per cent.

It is important to understand that both t_w and S are not trivially linked to the scaling properties of the inertial range fluctuations in the system. The value of t_w is dictated by the onset of some instability occurring at relatively large scales, whereas the avalanche of size S depends on the
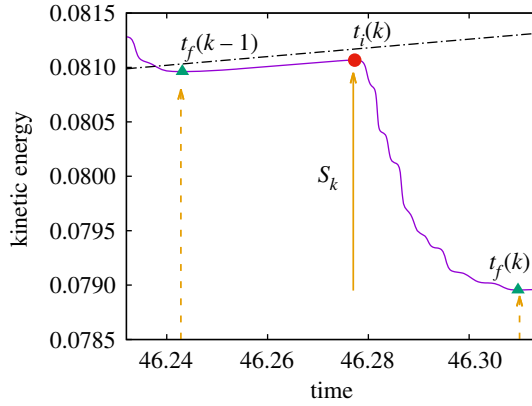
Figure 2. The figure illustrates our definition of event. We define an avalanche $S(k)$, $k = 1, 2, \ldots$, as the event in time for which $dE/dt < 0$ in the time interval $t \in [t_i(k), t_f(k)]$ where the times $t_i(k)$ and $t_f(k)$ correspond to the initial time and final time of the avalanche. The interevent time $t_w(k)$ is indicated by the two arrows: $t_w(k) = t_i(k) - t_f(k-1)$. (Online version in colour.)
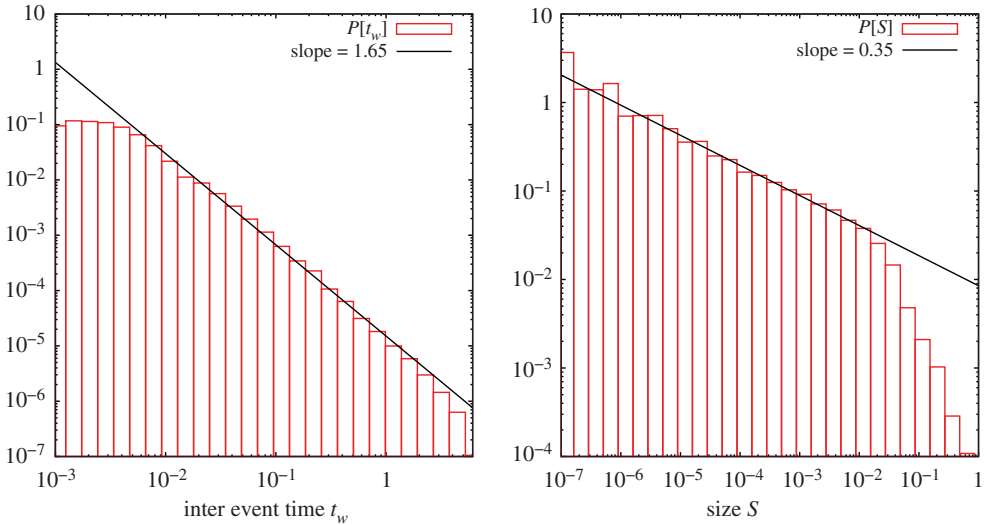

Figure 3. Probability distribution of the interevent time $P[t_w]$ (a) and avalanche size $P[S]$ (b) observed for modelling equation (2.1) with a constant power input. In both cases, well-defined scaling laws are observed with $P[t_w] \sim t_w^{-1.65}$ and $P[S] \sim S^{-0.35}$. The accuracy in the exponents is of the order of 5%. (Online version in colour.)

short intermittent bursts of energy dissipations. Although the initial and the final times of an avalanche occur when $\epsilon(t) > \phi$ and $\epsilon(t) < \phi$, respectively, its size does not necessarily correspond to large or small values of $\epsilon - \phi$. In other words, large values of ϵ can occur for both small and large values of the avalanche size S. More precisely, the size $S(k)$ of the avalanche can be computed from equation (2.3) to give $S(k) = \int_{t_i(k)}^{t_f(k)} \epsilon(t) \, dt - \phi(t_f(k) - t_i(k))$ with $\epsilon(t) \geq \phi$ for $t \in [t_i(k), t_f(k)]$. Upon denoting the event duration by $\tau(k) = t_f(k) - t_i(k)$, the probability distribution of $S(k)$ can be obtained from the probability distribution of $\tilde{\epsilon}(\tau(k)) \equiv \int_{t_i(k)}^{t_f(k)} \epsilon(t) \, dt$ constrained by the previously mentioned conditions on $\epsilon(t)$. Furthermore, $\tilde{\epsilon}(\tau(k))$ should be computed in the dissipation range and its fluctuations are correlated (with some non-trivial lag time depending on $\tau(k)$) to the probability distribution of the large-scale velocity fluctuations, i.e. to the kinetic energy. In summary, besides the fact that the statistical properties of $S(k)$ depend on the inertial range fluctuations in a complicated way, the probability distribution of $P[S]$ should be self-consistent with the fluctuations of the kinetic energy. Thus, the knowledge on the statistical properties of
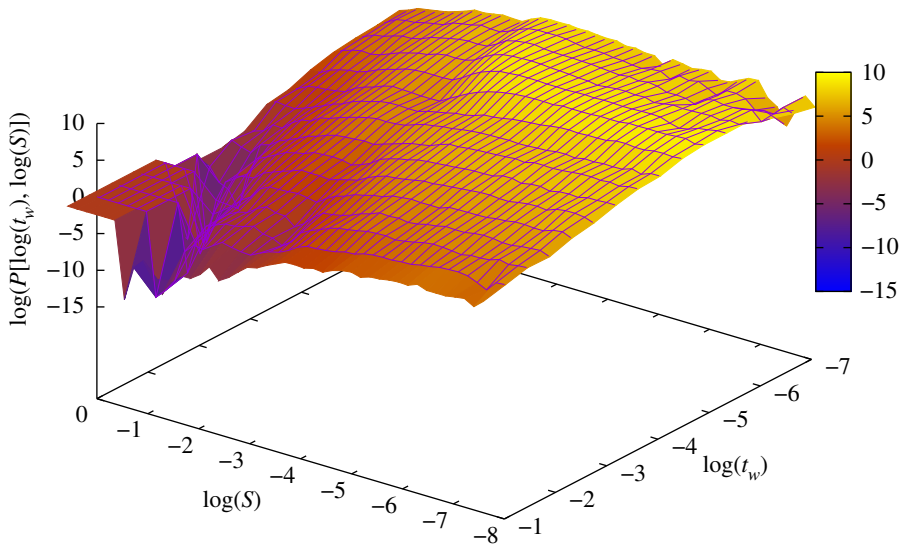
Figure 4. In the figure, we show a two-dimensional map for the PDF of both $\log(S)$ and $\log(t_w)$. The figure highlights the statistical independence of the avalanche size and the interevent times. By looking at the figure, there is no evidence of any correlation between the two variables, and by a numerical check, their correlation was found of the order of 7%. (Online version in colour.)

the inertial range velocity fluctuations does not provide any short cut to estimate the probability distributions of both t_w and S.

We shall see later (in §3(b)) that the statistical properties of both t_w and S depend on the way we force our system, i.e. on the physical mechanism of energy input in the system. From this point of view, both t_w and S may be considered as *bulk quantities* which characterize the random dynamics of the energy behaviour with respect to the (given) external forcing. There is no reason for *a priori* for $P[t_w]$ and/or $P[S]$ to be scaling functions of their arguments. The results shown in figure 3 are therefore non-trivial.

An important observation is that there exists no statistical correlation between t_w and S. In figure 4, we show the joint probability distribution of t_w and S which does not provide any significant hint of correlations between t_w and S. The statistical independence of t_w and S is also observed in other systems, such as amorphous solids and/or soft glasses, where avalanche dynamics are also characterized by scaling functions of $P[t_w]$ and $P[S]$, albeit with completely different scaling exponents [7,9]. Furthermore, a detailed analysis of seismic events [15] shows that no significant correlations exist between their interevent times and earthquake magnitudes. Thus, the statistical independence of t_w and S is a rather common feature observed in systems characterized by avalanche-like dynamics.

Following [6], we investigate a rather intriguing property of the interevent time statistics. Using our definition of avalanche, its size S spans from some minimum, say S_0, to some maximum S_M. Analogously, the interevent times t_w spans from some minimum t_0 and some maximum t_M. Let us now consider the interevent times t_w occurring for avalanches greater than λS_0, where λ is some real positive number. The interevent times are modified as illustrated in figure 5. The figure shows the energy behaviour $E(t)$ during a relatively short time window: the continuous line represents $E(t)$ and the red points highlight the avalanches occurring during the selected time window. One can see two relatively large avalanches occurring at the beginning and at the end of the time window with rather small events in between. Here large and small refer to the avalanche size. We also highlight the interevent times between avalanches. Once we consider avalanches bigger than λS_0 ($\lambda > 1$), for some value of λ, the two avalanches in the middle are neglected and the interevent time (corresponding now to the dashed line in the figure) becomes longer and
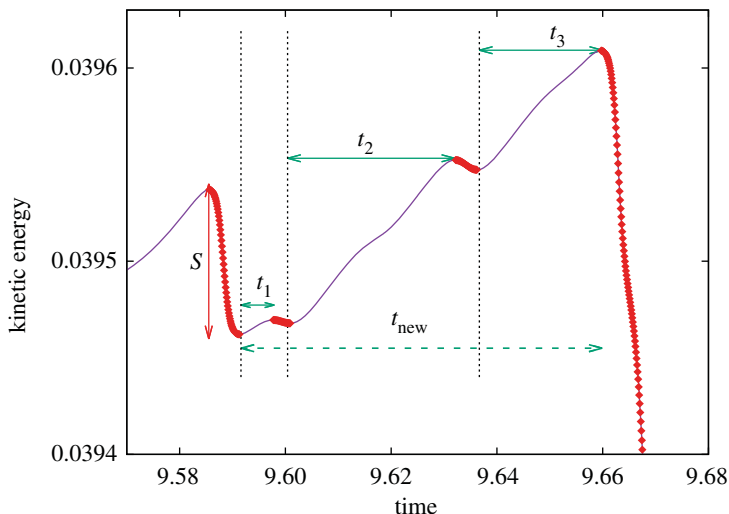
Figure 5. The figure illustrates the change in the quantitative definition of the interevent time by looking at avalanches whose sizes S are larger than some threshold S_{th}. The continuous line corresponds to $E(t)$. The red dots indicate when $dE/dt < 0$. We observe four avalanches. The vertical continuous arrow shows the size S of the first avalanche on the left. If we assume the two middle avalanches to be smaller than S_{th}, the interevent time between the first and the next avalanche becomes $t_1 + t_2 + t_3$. (Online version in colour.)

is approximatively equal, in this particular example, to $t_1 + t_2 + t_3$. Obviously, the probability distribution $P[S]$ remains unchanged and we may further assume that t_0 remains unchanged as well as S_M. However, clearly, this is not the case for t_M, which we expect to increase. Thus we should wonder how the probability distribution of the interevent times changes. We denote this new probability distribution by $P[t_w|S_{th}]$ where $S_{th} = \lambda S_0$ is the size of the avalanches disregarded for the computation of t_w.

In figure 6, we show $P[t_w|S_{th}]$ for different values of S_{th} spanning almost two orders of magnitude. A rather striking result is observed: $P[t_w|S_{th}]$ is invariant, i.e. it is a scaling function with the same exponent $\alpha \sim 1.65$. We notice that the very same results are observed in the interevent times of avalanches for amorphous materials [7,9] and earthquakes [6,10], although $\alpha < 1$ and the scaling range is smaller compared to what is observed in figure 6. In the model presented in this paper, the invariance of $P[t_w|S_{th}]$ is quite clear and striking. $P[t_w|S_{th}]$ can therefore be considered invariant with respect to the transformation $S_0 \to \lambda S_0$.

What is the physical meaning behind the results shown in figure 6? This is a non-trivial question which we will try to answer qualitatively. The scaling transformation $S_0 \to \lambda S_0$ should be considered equivalent to a kind of *coarse grained* transformation in the system: upon considering larger avalanche sizes we study the dynamics of the system on a *longer* time scale. Thus we may consider the invariance of $P[t_w|S_{th}]$ as the signature of a *scale invariance* of the system dynamics. This would be somehow trivial *if* $P[t_w|S_{th}]$ were a scaling function of both t_w and S_{th}. However, this is not the case in our system (nor in amorphous materials nor for earthquakes) simply because interevent times are statistically independent of the avalanche sizes. Thus, we are looking at a rather peculiar case of scale invariance, which deserves a deeper investigation. In §4 we make the first step in this direction.

(a) A different statistical analysis

First, we illustrate how the definition of $P[t_w|S_{th}]$ gives different results using different approaches usually employed in the analysis of intermittent or random processes. In particular, we consider a rather common approach based on the statistical properties of level-crossing. Given the dynamics
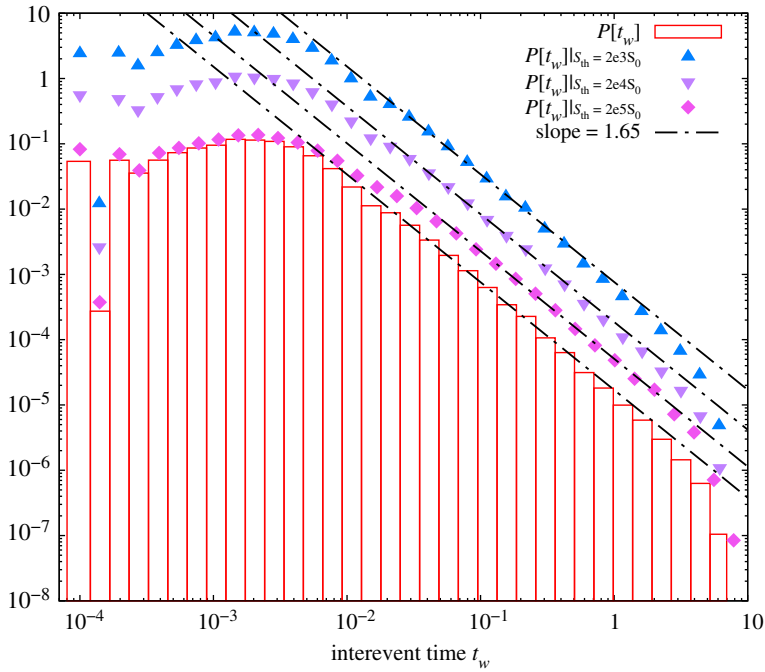
Figure 6. The figure illustrates the behaviour of the probability distribution $P[t_w|S_{th}]$ which is the probability distribution of the interevent times t_w occurring between avalanches of size $S > S_{th}$. Here we show the results for three different values of S_{th} which is defined with respect to the smallest size we found in our system, S_0, by $S_{th} = \lambda S_0$. Upon changing S_{th} for over two orders of magnitude the probability distribution of the interevent times preserves its scaling behaviour with the same scaling exponents as shown in figure 3. (Online version in colour.)

of energy dissipation $\epsilon(t)$ one can define an avalanche event as the time interval $[t_i(k), t_f(k)]$ for which $\epsilon(t) > \epsilon_*$. The size S of the event is defined as $S(k) = \int_{t_i(k)}^{t_f(k)} \epsilon \, \mathrm{d}t$ while the interevent times are defined as usual $t_w(k) = t_i(k) - t_f(k-1)$. This definition of event refers to *level-crossing* because it depends on ϵ_*. The statistical properties of level crossing are relevant quantities worth investigating in many physical and mathematical problems and there exists an extensive literature on the subject [16,17]. In the framework of level-crossing, both $P[S]$ and, more importantly, $P[t_w]$ depend on ϵ_*. We now use this approach using our model and the very same dataset as employed in figure 6. In particular, we focus on the probability distribution of the interevent times $P[t_w|\epsilon_*]$ for different values of ϵ_* in figure 7. Different to figure 6 $P[t_w|\epsilon_*]$ is no longer invariant, i.e. it is still a scaling function of t_w but with scaling exponents α which decrease upon increasing ϵ_*. We argue that the different results shown in figures 6 and 7 can be explained by noticing that upon increasing ϵ_* we are changing the physical meaning of the event. This is because in the ϵ based definition of size, the term $\phi(t_f(k) - t_i(k))$ is missing. Consider, for instance, a relatively large event selected with the level-crossing at some value ϵ_*. Upon increasing ϵ_* the initial time of the event is shifted and the corresponding interevent time increases or event duration decreases. This implies that t_w and S may acquire non-negligible correlations for large enough ϵ_*. On the contrary, the scale invariance shown in figure 6 is based on the same definition of the event regardless of the threshold S_{th}. It is true that the definition of event used in figure 6 is also based on a threshold, namely ϕ, however, looking at the interevent times for avalanches $S > S_{th}$ we retain the same definition (i.e. the events for which $\epsilon(t) \geq \phi$) while neglecting the computation of t_w avalanches smaller than S_{th}. Thus $P[t_w|S_{th}]$ and $P[t_w|\epsilon_*]$ refers to two different statistical properties of the system that are not related to one another. This example highlights the fact that the scale transformation $\epsilon_* \to \lambda \epsilon_*$ cannot be considered a *coarse-grained* transformation in the
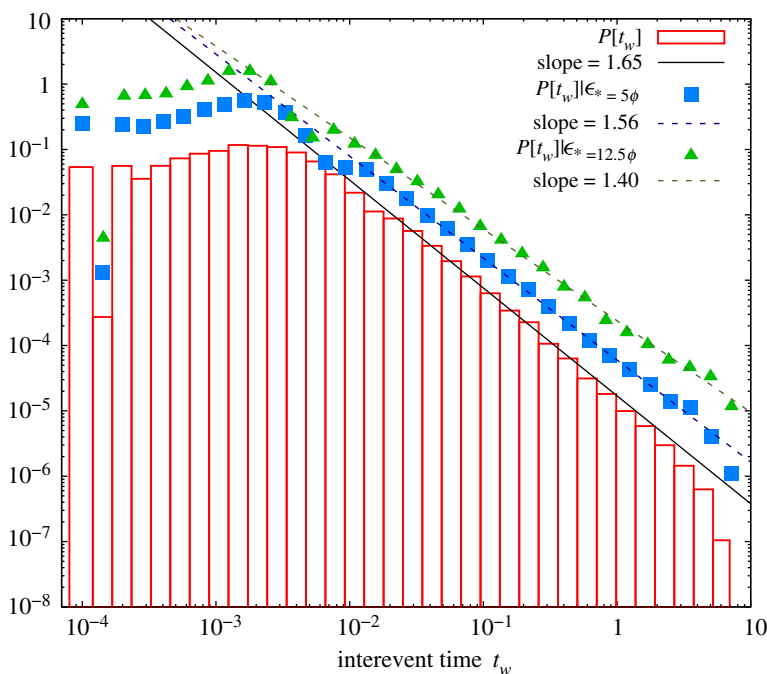
Figure 7. In this figure, we compute the probability distribution of $P[t_w, \epsilon_*]$ which represents the probability distribution of interevent times between consecutive events where $\epsilon(t) > \epsilon_*$ where $\epsilon(t)$ is the rate of energy dissipation and ϵ_* is defined with respect to the constant injected power $\phi = 4A$. Contrary to what we observe in figure 6, the probability distribution $P[t_w, \epsilon_*]$ is not invariant upon increasing ϵ_*. (Online version in colour.)

sense previously discussed. We are still looking at longer time scales but we are also considering different events. This is a crucial point often not properly taken into account in various statistical analyses.

Finally, let us remark that the results shown in figure 7 beautifully illustrate that the property of scale invariance shown in figure 6 is not trivially linked to the probability distribution of $\epsilon(t)$ which is the same for both cases.

(b) A different forcing

Another important issue to clarify is the relevance of the large-scale forcing. We have previously argued that the statistical properties of both t_w and S should be considered as *bulk* quantities of our system, i.e. quantities linked to the way the system is forced and describing the dynamics *with respect to the forcing mechanism*. We now considered exactly the same model given by equation (2.1) but we assume f_n to be a random Gaussian process δ correlated in time and acting for $n = 1$. We use the same definition of event size introduced in the first part of this section, namely $S(k) = -\int_{t_i(k)}^{t_f(k)} dt(dE/dt)$ and we look at the probability distribution of $P[t_w]$ and $P[S]$.

There is no evidence of $P[t_w]$ being a scaling function of t_w, see figure 8. Actually, we observe that $P[t_w]$ can be very well approximated by an exponential distribution. Notice that the scaling properties of the shell model, i.e. the anomalous scaling $\langle |u_n||^p \rangle \sim k_n^{-\zeta(p)}$, remain unchanged using the stochastic forcing. Thus, figure 8 illustrates the point raised at the beginning of this section, namely that the statistical properties of t_w and S are not linked to the inertial range dynamic, but a different forcing gives different statistical properties of the avalanche events. This also implies that scale invariance shown in figure 6 is not due to the scaling properties of the inertial range dynamics.
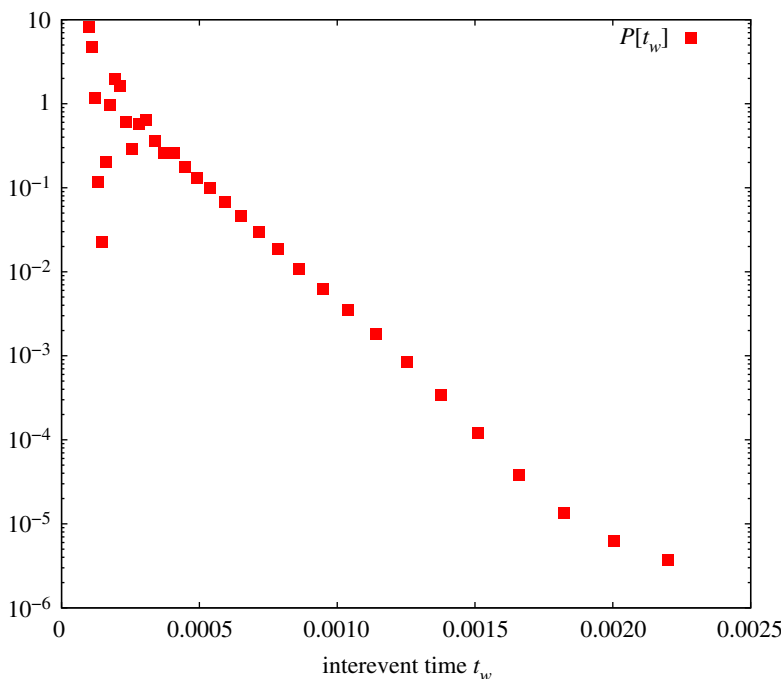
Figure 8. Probability distribution $P[t_w]$ of the interevent times t_w obtained for a random forcing acting on the shell 1. Notice that $P[t_w]$ does not show any scaling behaviour and it can be approximated as an exponential distribution. (Online version in colour.)

4. A tentative theory for scaling exponents

We will provide a theoretical framework to discuss the results illustrated in §3 and in particular the property of scale invariance of $P[t_w|S_{th}]$ shown in figure 6. To fix our theoretical analysis, we assume that the probability distributions of the avalanche size S and interevent time t_w are scaling functions of their arguments, i.e.

$$P[S] = \frac{Z_S}{S^\gamma} \quad S \in [S_0, S_M] \tag{4.1}$$

and

$$P[t_w] = \frac{Z_t}{t^\alpha} \quad t_w \in [t_0, t_M], \tag{4.2}$$

where Z_S and Z_t are normalization factor. Note that we focus only on the scaling part of $P[t_w]$ and $P[S]$ neglecting regions where no scaling is observed. For our analysis this approximation is reasonable. Upon assuming $t_M \gg t_0$ and $S_M \gg S_0$, for $\gamma < 1$ and $\alpha > 1$ (as in our case) we have to the leading order $Z_S = S_M^{\gamma-1}$ and $Z_t = t_0^{\alpha-1}$.

We now consider the quantity:

$$X \equiv \frac{E_{stored}}{E_{released}}, \tag{4.3}$$

where E_{stored} is energy input in the system during the interevent time and $E_{released} = S$ is the energy released during an avalanche. Because we are forcing the system with a constant power input $\phi = 4A$, E_{stored} is well approximated by the quantity $4At_w$. Hereafter, we can disregard the factor $4A$ and we write

$$X = \frac{t_w}{S}, \tag{4.4}$$

where X is a random variable that describes the fluctuations of the dynamical process in the energy behaviour of the system, namely the stored energy in the system with respect to the energy

released. Now, let us consider the scale transformation

$$S_0 \to \lambda S_0. \tag{4.5}$$

As we argued in §3 , the scale transformation (4.5) does not change t_0 whereas S_M is not changed by definition. Because we expect t_M to increase, we can write

$$t_M \to \lambda^H t_M, \tag{4.6}$$

where H is yet unknown. In general we expect that $P[t_w]$ will change because of the scale transformation (4.5) and the exponent α may become a function of λ, i.e.

$$P[t_w] = \frac{t_0^{\alpha(\lambda)-1}}{t_w^{\alpha(\lambda)}}. \tag{4.7}$$

We are interested in the probability distribution $P[X]$. Under the scale transformation (4.5), we expect that X and $P[X]$ depend on λ. The scale transformation (4.5) can be interpreted as a 'coarse-grained' transformation following our discussion in §3. From this point of view, scale invariance in the system should be equivalent to saying that $P[X]$ does not depend on λ. Because t_w and S are independent variables, we can easily compute the moments $\langle X^{-n} \rangle$ and $\langle X^n \rangle$ for any n, where $\langle .. \rangle$ is the average over $P[X]$. Using (4.1) and (4.7) we obtain:

$$\langle X^{-n} \rangle = \left[\frac{S_M}{t_0} \right]^n. \tag{4.8}$$

Since neither t_0 nor S_M depends on λ, we obtain that $\langle X^{-n} \rangle$ is independent of λ. To simplify the following computation, we can assume $t_0 = 1$ and $S_M = 1$ without loss of generality. To find $\langle X^n \rangle$ with $n > 0$, a little algebra gives:

$$\langle X^n \rangle = t_M^{n-\alpha(\lambda)+1} S_0^{1-n-\gamma}. \tag{4.9}$$

Under the scale transformation (4.5), we then obtain

$$\langle X^n \rangle \to \langle X^n \rangle \lambda^{[n(H-1)+1+H-H\alpha(\lambda)-\gamma]}. \tag{4.10}$$

For scale invariance to hold, we must require that for any $n > 0$ the following equation is satisfied:

$$n(H - 1) + 1 + H - H\alpha(\lambda) - \gamma = 0.$$

This implies that $H = 1$, $\alpha(\lambda)$ is independent of λ and:

$$\alpha + \gamma = 2. \tag{4.11}$$

This tells us something interesting: first of all, the scale invariance of $P[X]$ is equivalent to the scale invariance of $P[t_w|S_{\text{th}}]]$ under the scale transformation (4.5) and α does not depend on λ; secondly, we obtain a non-trivial result relating the scaling exponents γ and α and expressed by equation (4.11). It is important to note that our results do not 'prove' scale invariance in our system. What we can prove is that *if the statistical properties of the system are scale-invariant with respect to* (4.5) *then the scaling exponents α and γ are not independent and satisfy equation* (4.11). The numerical values of α and γ obtained in the previous section, see figure 3, are in excellent agreement with (4.11) within a few per cent. As a side product of equation (4.11), we observe that the probability distribution of $\varXi = 1/S$ should be a scaling quantity with scaling exponent α. Using this observation, we can provide a direct test of equation (4.11) by comparing the probability distribution of $1/S$ with respect to the probability distribution $P[t_w]$. This is done in figure 9 which shows an excellent agreement with (4.11). Thus, besides the numerical estimate of α and γ, we can check the validity of equation (4.11) directly in figure 9.

It is tempting to investigate whether the same argument can be applied to other systems. In a soft glass and/or in an amorphous solid driven by a constant shear rate $\dot{\varGamma}$, the internal energy grows as $\sigma \varGamma$ where σ is the internal stress and $\varGamma = \int \dot{\varGamma} \, dt$ is the applied strain. Since $\sigma \sim \varGamma$ the internal energy of the system, during the time between two consecutive avalanches, grows as t_w^2
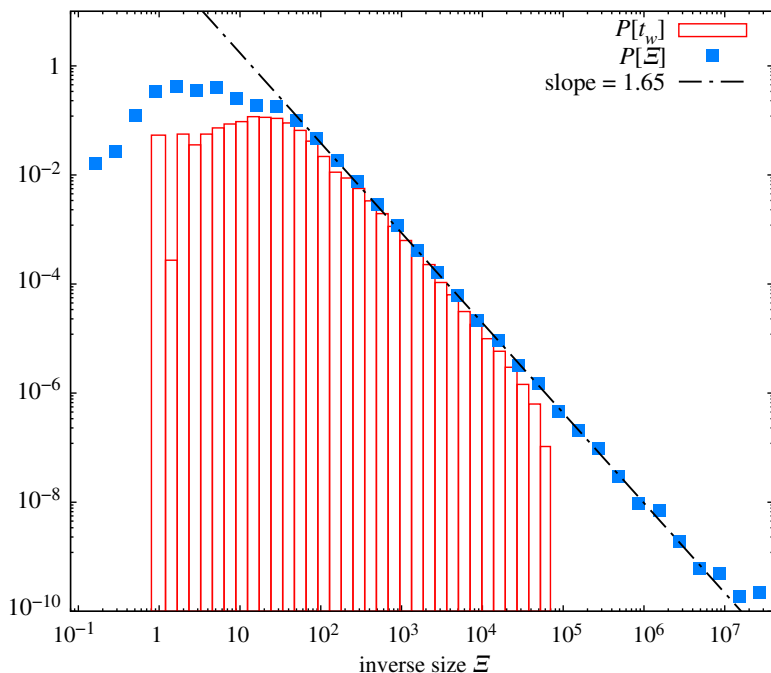
Figure 9. Quantitative check of our theoretical approach. According to our theoretical interpretation of scale invariance, the probability distribution of $\Xi \equiv 1/S$ should show the scaling properties of the interevent time probability distribution $P[t_w]$ (here conveniently shifted). This is checked in the figure where we compute $P[\Xi]$. (Online version in colour.)
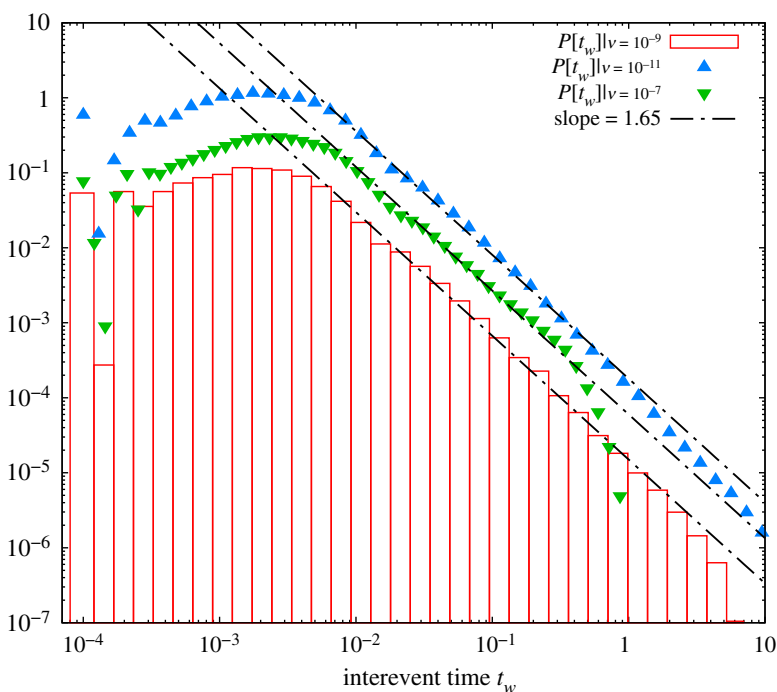

Figure 10. Probability distribution $P[t_w]$ of the interevent times t_w obtained equations (2.1) and (2.2) for different values of ν. Upon changing the *Re* number, the scaling properties of the $P[t_w]$ does not change. (Online version in colour.)

under the (usual) assumption that $\dot{\Gamma}$ is constant. For the systems where both γ and α have been measured [7,9] ($\alpha < 1$ and $\gamma \sim 1.33 > 1$), the same reasoning introduced above in this section gives:

$$\langle X^{-n} \rangle = \left[\frac{t_M^2}{S_0} \right]^n \tag{4.12}$$

and

$$\langle X^n \rangle = \left[\frac{S_0}{S_M} \right]^{\gamma - 1} S_M^n \left[\frac{t_M}{t_0} \right]^\alpha \frac{1}{t_M t_0^{2n-1}}. \tag{4.13}$$

Equation (4.12) tells us that upon scaling $t_M \to \lambda^{1/2} t_M$ (i.e. $H = 1/2$) for scale invariance to hold. Equation (4.13) then implies that:

$$\alpha = 3 - 2\gamma. \tag{4.14}$$

For $\gamma \sim 1.33$ we obtain $\alpha \sim 0.33$ in very good agreement with the numerical and experimental results discussed in [7]. Thus, it seems that our argument can be considered rather general and independent of the detailed physical mechanisms behind the avalanche dynamics. Notice that in amorphous solids, and in particular in the framework of a lattice mesoscopic elasto-plastic model [3], the exponent γ characterizing the scaling of $P(S)$ has been related in [18] to the probability distribution $P(x)$ of the local distance x to instabilities. Then, equation (4.14) suggests that from the knowledge of $P(x)$, one should be able to compute t_w. How this may occur is an interesting yet open question.

The common assumption in deriving equations (4.11) and (4.14) is that the system is scale-invariant under the transformation (4.5). Physically, in both cases, we can argue that the system approaches some kind of *critical dynamics* where two different phases (laminar/turbulent in our case) or (no flow/flow for amorphous systems) are dynamically competing. The scale invariance, *if it occurs*, then implies that the scaling exponents of avalanche size and interevent times are linked. This is a rather non-trivial outcome of our analysis.

It is equally tempting to address the case of earthquakes where the Gutenberg-Richter scaling implies $\gamma = 5/3$. For earthquakes, however, we have no idea how stored energy depends on the time between two consecutive events. For $\gamma = 5/3$ we have $\alpha = 0.33$ using (4.11) in agreement with the results obtained in [6] and, interestingly, close to the one obtained in amorphous solids. This suggests that the stored energy grows linearly in time. How reasonable is this? We can speculate that energy stored is still given by $\sigma \Gamma$ as in amorphous materials. Then, following [19] we can assume that σ is equal to the so-called *apparent stress*, which is commonly assumed to be constant. The strain Γ is due to tectonic motion and, in a very first approximation, we may argue that it is proportional to time. Using these (strong) assumptions on the stress and the strain, we can write $X \sim t_w/S$ and we can repeat the same reasoning leading to (4.11), which leads of course to the same relation $\alpha + \gamma = 2$. Although exciting, we should consider our finding very preliminary and very speculative and the argument definitively deserves more rigorous investigations.

5. Conclusion

This paper discusses avalanche dynamics in a shell model of turbulence forced with a constant power input ϕ. The avalanche-like events are characterized by a sharp negative decrease of the kinetic energy E. We investigated two relevant statistical properties, namely the probability distribution $P[S]$ of the avalanche size S, corresponding to the energy drop during an event, and the probability distribution $P[t_w]$ of the interevent time t_w between two consecutive avalanches. Both probability distributions show a clear scaling behaviour $P[S] \sim S^{-\gamma}$, $\gamma = 0.35$, and $P[t_w] \sim t_w^{-\alpha}$, $\alpha = 1.65$. We have provided numerical evidence that the probability distribution of t_w shows *scale invariance*: upon computing t_w between events of size $S \geq S_{\text{th}}$, $P[t_w]$ shows the same scaling behaviour independently of S_{th}, while importantly t_w and S are statistically independent variables.

This scale invariance is similar to what has been observed in amorphous materials and in the analysis of earthquake catalogues. Assuming scale invariance to hold, we provide a

simple theoretical argument stating that the scaling exponents α and γ must satisfy the relation $\alpha + \gamma = 2$ in excellent agreement with the numerical results. We have generalized our approach for amorphous materials with very good agreement against numerical and experimental results, giving $\alpha + 2\gamma = 3$. Relations (4.11) and (4.14) are here derived for the first time. Once again, we remind the reader that, in our view, both relations are consequences of scale invariance in the system, and they do not necessarily hold for all systems showing avalanche-like dynamics. For instance, in [7] it was shown that the statistical properties of t_w in amorphous materials depend critically on the material stiffness. In particular, scale invariance is observed for very 'rigid' systems, whereas this is not true for softer materials. When scale invariance holds, then the scaling exponents of $P[t_w]$ and $P[S]$ are related to one another. This is something new and somehow unexpected.

One important point highlighted in the paper is that neither $P[S]$ nor $P[t_w]$ can be obtained using statistical properties of inertial range (intermittent) fluctuations. For this reason, we consider both $P[S]$ and $P[t_w]$ as *bulk quantities* related to the forcing mechanism. An important possible question to address is whether the scaling exponents depend on the *Re* number.

We numerically checked that neither α nor γ are functions of *Re*. In figure 10, we show $P[t_w]$ for $\nu = 10^{-7}$, 10^{-9} and 10^{-11}: upon changing *Re* by 4 orders of magnitude, the scaling exponent α does not change (the same is true for γ, not shown). The only relevant change in $P[t_w]$ concerns the range where scaling is observed which seems to decrease with *Re*.

Another important point is how to properly define scale invariance for the probability distribution $P[t_w]$. Interevent times (sometimes referred to as return times or waiting times) are important statistical variables discussed for many theoretical frameworks in physics. Given a random (intermittent) process $v(t)$, one can study $P[t_w, L]$ as a function of the time at which $v(t)$ is larger than some level L. In the case discussed in this paper, $P[t_w, L]$ does not show any scale invariance. In fact, upon changing L, we are also changing our definition of *event size*. Our approach, following [6], is to fix the definition of event and then disregard it in the computation of t_w events of size larger than some threshold S_{th}: we are looking at some longer time for the *same* set of events. One may argue that this is a minor detail, but we have clearly shown that this is not the case in our system and we argue the same is true for other physical systems.

There is no *a priori* reason to assume that the interevent time distribution $P[t_w]$ is scale-invariant in the sense discussed in this paper (see also the discussion at the end of §4 and [7]). However, it is a remarkable result, here presented for the first time, that if scale invariance holds then the scaling exponents of size and interevent time distributions must be related.

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