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Global seismic tomography: the inverse problem and beyond

J Trampert

Department of Geophysics, Utrecht University, PO BOX 80021, 3508 TA Utrecht, the Netherlands

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Abstract. Global seismic tomography has produced a great amount of robust information concerning the three-dimensional extent of the Earth's internal structure. This has stimulated a multidisciplinary discussion aimed at understanding the mechanisms which govern the internal evolution of our planet. A brief overview of seismic tomography is presented. Since geodynamical understanding is the main purpose of seismic tomography, some suggestions are made on how to evolve from a predominantly qualitative to a more quantitative interpretation of its results. We argue that without a more systematic and realistic error and resolution analysis, interpretations might be misleading. Assuming a steady increase of data quality and coverage, the most challenging aspect of seismic tomography will be to take the nonlinearity of the problem fully into account. It is hoped that this contribution stimulates some discussion in that direction.

1. Introduction

Global seismic tomography has been a very active field of research since the first systematic efforts in the early 1980s. Over the years several good and exhaustive reviews have been written on the subject (e.g. Dziewonski and Woodhouse 1987, Woodhouse and Dziewonski 1989, Masters 1989, Romanowicz 1991, Montagner 1994, Masters and Shearer 1995, Ritzwoller and Lavely 1995, Dziewonski 1996). I will not try to reiterate the state-of-the-art in seismic tomography, but rather the opposite. My concern will be to point out some shortcomings in our modelling techniques which need to be overcome in order to advance from a predominantly qualitative to a more quantitative interpretation of its results. It is thus hoped to stimulate a discussion among seismologists and scientists from other fields with different experiences and approaches to yield new developments.

I will give a general overview of the techniques used in global seismic tomography. The most easily understandable case of travel-time tomography will be used to illustrate some specific points. The given summary is by no means exhaustive and as a consequence, the reference list far from complete. The main concern here is to convey the general ideas of seismic tomography to the nonspecialist in that field. Before beginning a detailed discussion, I would like to put seismic tomography in a more geodynamical context and point out that it is extremely encouraging to see that a great amount of overlapping information is emerging from different tomographic studies using different data, and/or different theories.

Seismic tomography is by far the most powerful tool to probe the Earth's deep interior. Seismic waves from big enough earthquakes can be observed throughout the world, and as they travel deep through the Earth or along the surface, their arrival times and shapes are impregnated with information on the medium they travelled through. The inverse

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problem in seismic tomography consists of mapping the Earth's three-dimensional elastic velocity field from large quantities of arrival times, body or surface waveforms and free oscillations. A recent quantitative comparison of many different studies (Ritzwoller and Lavely 1995) showed that a robust overlap of information concerning the Earth's structure was recovered using different data and different mapping strategies. For the upper mantle (figure 1, top panel) the main results are low-velocity zones associated with mid-oceanic ridges and regions of the western Pacific characterized by back-arc volcanism. These areas may be understood in terms of upwellings of hot material. High velocities are mainly continental areas (in particular shields) assumed to be compositionally different, cooler and of higher viscosity than the average upper mantle. At shallow depths in oceans, where the lithosphere is older, cooling is responsible for the creation of a thicker and hence faster than average lithosphere. This correlation with structures identified by plate tectonics is thus to be expected if we assume that low and high velocities mainly correspond to material hotter, respectively colder, than average. In the lower mantle (figure 1, bottom panel) the amplitude of heterogeneity is much smaller than in the upper mantle. Fast velocities are concentrated on a ring surrounding the Pacific. This ring roughly corresponds to the projection into the lower mantle of zones of convergence from surface tectonics and subducting slabs. This could be an indication of subducting slabs penetrating into the lower mantle, as shown in a more spectacular way by Grant (1994) and Van der Hilst *et al* (1997). Zones of low velocities are correlated with geoid highs and the positions of most world hotspots. It is hoped that these three-dimensional velocity models will provide a solid basis for understanding the driving forces of plate tectonics.

Seismic tomography maps the current thermodynamic and compositional state of heterogeneity in the flowing mantle and thus imposes severe constraints on possible models of convection in the mantle (Tackley *et al* 1994). The thickness of continental roots, the depth extent of the mid-ocean ridge signal and the change of lithospheric velocity versus age (Su *et al* 1992, Woodhouse and Trampert 1996) give important clues on the formation and evolution of the continental and oceanic lithosphere. The relative variation of P-wave velocities versus S-wave velocities puts strong mineralogical constraints on the composition of the mantle (Robertson and Woodhouse 1996). The thermal state of the lower mantle puts some boundary condition on possible geodynamo models explaining the Earth's magnetic field (Olsen and Glatzmaier 1996). It has long been recognized that there is a correlation between the geoid and seismic models (Hager *et al* 1985), and therefore both types of information (gravity and seismological) may give access to the three-dimensional density variations within the Earth. Seismic tomography thus fuels a strong interdisciplinary discussion in Earth sciences. The fundamental question which still awaits to be addressed in detail is: how well resolved and how accurate are our current tomography models? The answer to this question, however, is of primary importance for a quantitative interpretation.

The easy access to digital seismological data, the increasing computational facilities and the availability of robust matrix solvers now make it relatively straightforward to produce a tomographic model. Trying to make an interpretation of the model to understand the inner workings of the underlying real Earth is far more difficult. Many hidden problems make a straightforward interpretation more difficult than commonly assumed. For instance, there is no doubt that seismic tomography can detect subducting slabs, but it is nowhere near as clear whether or not it can answer the question of the depth extent of the penetration, owing to insufficient analyses of depth resolution and a lack of detailed understanding of the sensitivity of seismic velocity to changes in composition and thermodynamic parameters such as temperature and pressure. The answer to this latter question, however, is of primary importance for determining the exact nature of mantle convection. The spectrum

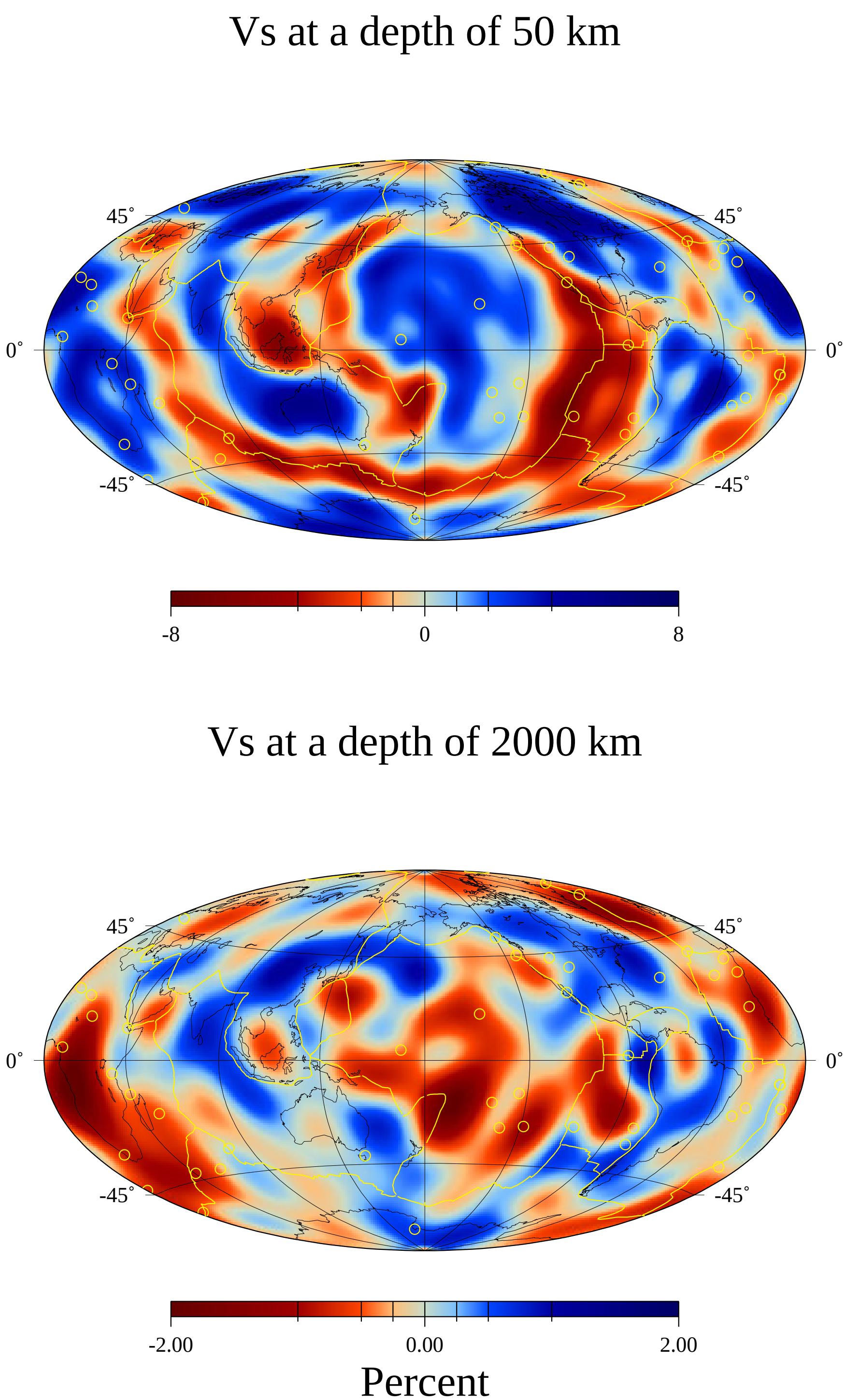

Figure 1. Maps of S-wave velocity perturbations of model S16RLBM (Woodhouse and Trampert 1996) at two different depths. The perturbations are in per cent relative to a reference model. Plate boundaries and hotspots are shown in yellow.

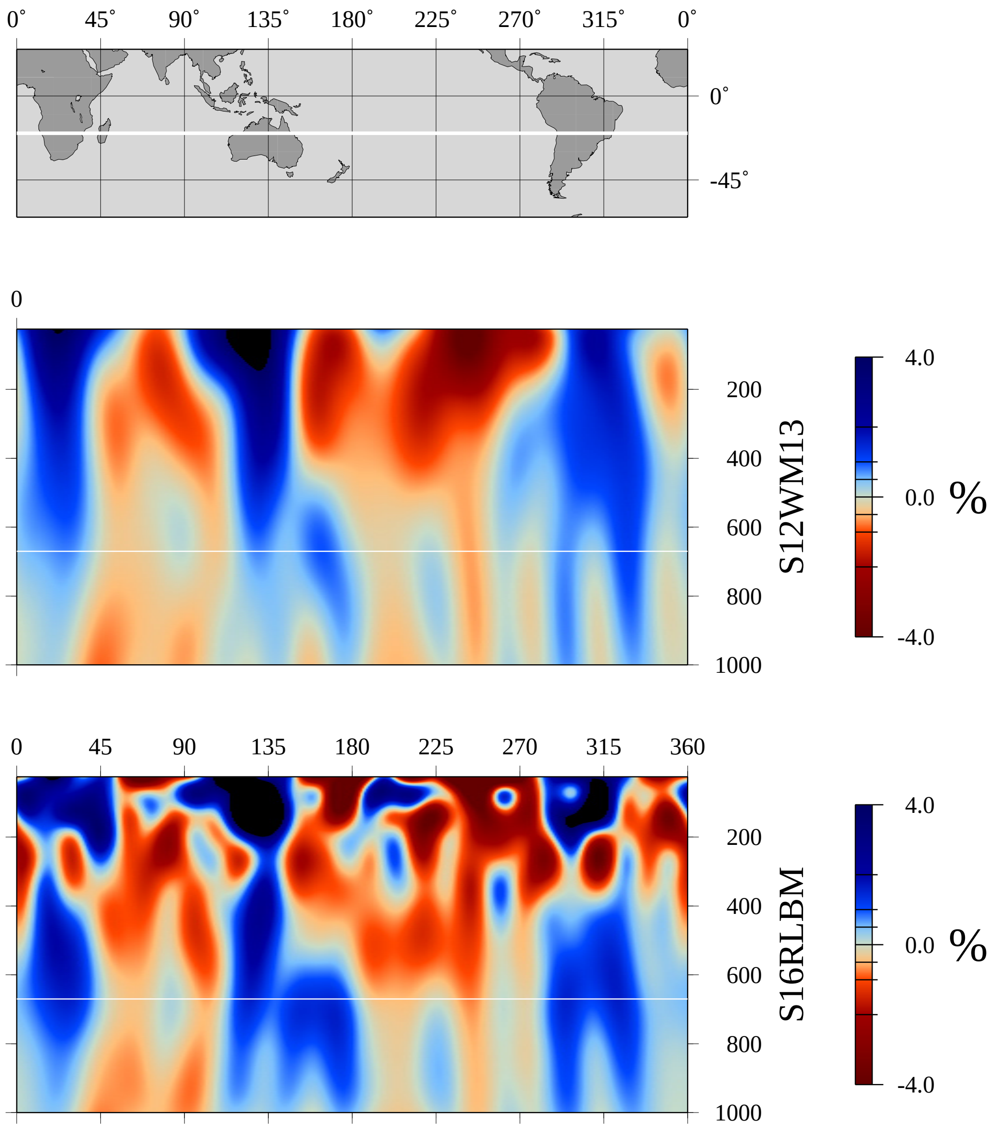

Figure 2. Cross sections of two different models at 20° South as shown in the top panel by the white line. The perturbations of S-wave velocities for model S16RLBM (Woodhouse and Trampert 1996) and model S12WM13 (Su *et al* 1994) are in per cent relative to the same reference model. The horizontal distances are in degrees of longitude and depths are in kilometres. The white lines in the two bottom panels divide the Earth's interior into the upper and lower mantle. This division corresponds to a strong discontinuity of average seismic velocities at that depth.

of heterogeneity is another question which has received a lot of attention in the literature. For a long time, arguments were put forward in favour of the idea that the spectrum is red (e.g. Su and Dziewonski 1991), meaning that the Earth's internal structure is dominated by long wavelengths. A few isolated studies pointed out that this might not be the case. Gudmundsson *et al* (1990) analysed P-wave travel-time residuals in a statistical manner and concluded that the Earth's spectrum is much whiter than had previously been assumed. Snieder *et al* (1991) showed that part of the observed long-wavelength features could be due to artefacts of low-pass filtration of much narrower structures, such as mid-oceanic ridges or subducting slabs. A quantitative analysis of small-scale structure leaking into long-wavelength structure has been given by Trampert and Snieder (1996). Furthermore, regularization of any kind, always necessary in the tomographic inverse problem, favours a red spectrum, and so does the use of integral data (Mochizuki 1993, Passier and Snieder 1995) which is always the case in seismic tomography. The nature of the spectrum, important for discriminating between many competing convection models, certainly needs to be re-examined. These are just a few examples to try to illustrate the difficulties arising from the interpretation of seismic tomography. To address these points, we suggest that we have to look in far more detail, than has been done up to now, at issues such as how accurate our models are (error estimation), how many different models are compatible with the data (nonuniqueness and resolution) and how to use more elaborate inverse techniques to take the inherent nonlinearity of the problem into account.

2. The inverse problem

2.1. Definition and general characteristics

Seismic tomography is mainly concerned with reconstructing the three-dimensional velocity field inside the Earth from observations of elastic waves at the surface. The forward problem consists of predicting a seismogram $s(t, \Delta)$, or parts of a seismogram, at a given distance Δ from the seismic source as a function of time t assuming a certain velocity field $v(r)$. Formally we have

$$s_i(t, \Delta) = \int_{\Omega[v(r)]} g_i[t, \Delta, v(r)]\, \mathrm{d}r \tag{1}$$

where r labels the position inside the Earth and i denotes a particular seismogram. g describes the physical theory of elastic wave propagation and Ω is the path of the wave (line, surface or volume). Nonlinearity may enter equation (1) in two ways: explicitly through the expression of g and implicitly through the path Ω which depends on $v(r)$. In seismology, all data are integral or average measurements of medium properties $v(r)$ along the path Ω. The inverse problem consists of finding the velocity field $v(r)$ from many different observations $s(t, \Delta)$.

The nature of the tomographic inverse problem is characterized by an uneven distribution of sources (earthquakes) and receivers (seismic stations on continents and oceanic islands only) resulting in an uneven sampling of the medium by elastic waves. This means that some parts of the Earth's interior are overdetermined while other parts remain underdetermined. The inverse problem is ill-posed under these conditions. Usually, its eigenvalue spectrum is falling off quite steeply which means that small errors in the data may produce large variations in the solution. The problem is said to be ill-conditioned. Both, ill-posedness and ill-conditioning go hand in hand with large nullspaces, which imply nonunique solutions. The answer to these problems is implicit or explicit regularization. Regularization can either be seen as reducing the possible model space or choosing one particular solution out of many possible ones. A last concern is that inverse mapping should be a consistent operation, meaning that the solution itself is independent of the parametrization used to describe the solution. Trampert and Lévêque (1990) illustrate inconsistency with a purely underdetermined problem with two parameters and one data point: $a + b = 2$. The least-squares solution is $a = 1$ and $b = 1$. If we change the parameters into $c = a + b$ and $d = b$, we find $c = 2$ and $d = 0$ which yields $a = 2$ and $b = 0$. Clearly, both solutions are correct but would give completely different interpretations. To understand what is happening, we have to evoke implicit regularization used by the least-squares algorithm itself. This little example shows that comparing solutions which used different parametrizations may prove difficult if inconsistent mapping techniques are used. An extensive discussion of this particular problem may be found in Tarantola (1987).

The geophysics community has been very active in inverse theory and as a result many textbooks describe in detail the theory as well as the practical side of specific techniques used in that field (e.g. Tarantola 1987, Menke 1989, Parker 1994, Sen and Stoffa 1995). In general, relation (1) is far too complicated to be of any practical use, so that simplifications in the expression of g, Ω and v need to be sought. As a result, most global tomography problems are solved using a linearized relationship between the data and model parameters, and a more or less complicated cost function is minimized in a least-squares sense to derive a linear operator which maps the data into an estimated solution. To be more specific we will need to discuss in more detail the different ingredients entering the inverse problem.

2.2. Data

A typical seismogram (figure 3) shows packets of P and S body waves and dispersed surface waves, which are the result of the seismic source, the elastic properties of the Earth through which the waves propagate and the characteristics of the seismic recorder. The nomenclature of seismic body waves is a code which describes their path through the Earth. The most basic information in a seismogram is the arrival times of the different waves. Travel-time tomography uses this simplified information for building Earth models. Alternatively, the dispersed surface waves or the whole seismogram may be used in tomography needing a more elaborate theory of wave propagation. In the case of travel times, the viewpoint of a propagating wave is commonly adopted, while in whole waveform modelling, the standing wave description (normal modes) is more frequently used.

Seismological data spanning more than three orders of magnitude in frequency have been used to map the structure of Earth's interior. They vary roughly from one second body-wave arrival times to normal mode periods of several thousand seconds. The first ones are sensitive to Earth structure at lengthscales as short as tens of kilometres, while the latter ones have wavelengths up to the Earth's radius itself. Intermediate frequency data consist mainly of whole body or surface waveforms. These different data sets taken from various parts in the seismogram sample the Earth's structure in different ways and give complementary information on the medium they travelled through.

2.3. Forward theory

The propagation of seismic waves in an elastic medium is governed by the elastodynamic equations. Assuming that the source excitation and the instrument response is known, this leads to a nonlinear relationship between the observed data and the given Earth model via the forward theory containing the physics of wave propagation. A full forward theory would allow us to model all the wiggles seen on figure 3, but would be excessively complicated. We therefore seek simplifications in g and Ω in expression (1). The simplest form of seismic tomography uses arrival times of body waves. In the ray theoretical approximation, the travel time T_i of a ray i along its path L_i (which is a line) is the integrated slowness $v(r)^{-1}$

$$T_i = \int_{L_i} \frac{\mathrm{d}r}{v(r)}. \tag{2}$$

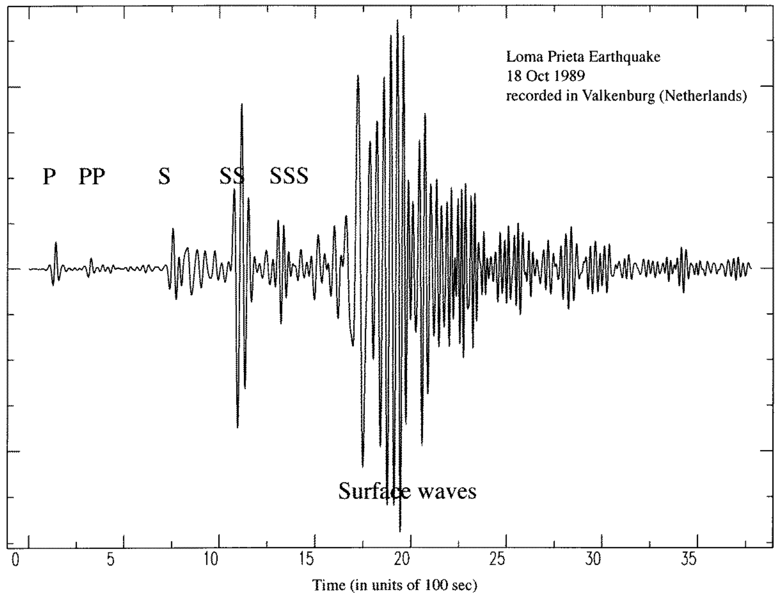

Figure 3. Example of a seismogram recorded at a distance of about 9000 km from the earthquake source. The body waves appear as relatively impulsive wavepackets whereas the surface waves are well dispersed in time.

We may express this as a travel-time difference, or delay time, with respect to a travel time in a reference Earth:

$$\delta T_i = T_i - T_i^0 = \int_{L_i} \frac{\mathrm{d}r}{v(r)} - \int_{L_i^0} \frac{\mathrm{d}r}{v_0(r)}. \tag{3}$$

Fermat's principle states that the travel time of a ray is stationary for small changes in ray location. This allows us to substitute the unknown raypath in the true Earth by the raypath in the reference Earth, introducing a second-order error and we finally arrive at

$$\delta T_i = -\int_{L_i^0} \frac{\delta v(r)}{v_0(r)^2}\, \mathrm{d}r. \tag{4}$$

This equation expresses a linear relationship between the observed delay times and the perturbations $\delta v(r)$ to the reference Earth.

In a similar way, Rayleigh's principle is used to derive a linearized relationship between three-dimensional perturbations from a radially symmetric reference model and the resulting perturbations in the dispersed surface-wave data. In the more general case of full wave-form inversion, perturbation theory with some form of Born approximation is used to calculate the necessary partial derivatives. All global tomography studies to date use a linearized forward theory which for a particular measurement $\delta d_i = s_i - \int g_i(v_0)\, \mathrm{d}r$ can formally be written as

$$\delta d_i = \int G_i(r) \delta v(r)\, \mathrm{d}r \tag{5}$$

where $G_i(r)$ are the partial derivatives about $v_0(r)$ of the nonlinear functional $g_i[t, \Delta, v(r)]$ of expression (1) or data kernels at position r inside the Earth.

2.4. *Parametrization of the model*

In a strict sense the models $\delta v(r)$ we want to infer from the data are continuous functions of position. This means of course that our problem is infinite dimensional in the model space. We now seek simplifications of v in equation (1) and it is convenient to expand the model in a complete set of basis functions $B_j(r)$, most appropriate for the given problem:

$$\delta v(r) = \sum_{j=1}^{\infty} m_j B_j(r). \tag{6}$$

Many different choices of basis functions are possible. The most common choices in global tomography are blocks and spherical harmonics (together with polynomials at depth). To describe the model fully, the chosen set has to be complete (the summation in equation (6) is carried out to infinity), and then all parametrizations are equivalent. For practical reasons and the limited resolution of the data, we have to choose an upper limit L, however. This leads then to a classical linear inverse problem for L coefficients m_j which may be represented by the matrix equation

$$\delta \boldsymbol{d} = \mathbf{A}\boldsymbol{m} \tag{7}$$

where the matrix $\mathbf{A}$ is defined by $A_{ij} = \int G_i(r) B_j(r)\, \mathrm{d}r$. The truncation of the expansion leads to a smoothed estimation of the true model, regardless of the real smoothing properties of the data. It is important to realize that the truncation implies implicit regularization of the inverse problem. Furthermore, different choices of basis functions (e.g. rough edges of blocks versus smooth edges of spherical harmonics) also lead to different implicit regularizations. Consistency problems in the chosen inverse operator may make comparisons of studies with different choices difficult. An additional problem is that truncated basis functions may leak into the solution, and give a biased estimation of the model (Trampert and Snieder 1996).

2.5. *Cost function*

The data $\delta \boldsymbol{d}$ and model parameters $\boldsymbol{m}$ are normally assumed to be elements of metric spaces $\mathcal{D}$ and $\mathcal{M}$, respectively. Solving the inverse problem is giving a rule for mapping an element of $\mathcal{D}$ into an element of $\mathcal{M}$. The global seismic tomography problem is usually not well-posed, and hence the desired mapping rule is not uniquely defined. We need to have some criteria which choose one particular mapping. A general way of introducing such a criterion is to define a cost function of the form

$$C_\lambda = \Delta_{\mathcal{D}}(\delta \boldsymbol{d}, \mathbf{A}\boldsymbol{m}) + \lambda \Delta_{\mathcal{M}}(\boldsymbol{m}, \boldsymbol{m}_0). \tag{8}$$

$\Delta_{\mathcal{D}}$ and $\Delta_{\mathcal{M}}$ are metrics measuring the distance between observed and predicted points in the data space and between a given model and a reference point in the model space, respectively. In the case of a Bayesian viewpoint, $\Delta_{\mathcal{D}}(\delta \boldsymbol{d}, \mathbf{A}\boldsymbol{m})$ would have the form $(\delta \boldsymbol{d} - \mathbf{A}\boldsymbol{m})^\dagger \mathbf{C}_d^{-1} (\delta \boldsymbol{d} - \mathbf{A}\boldsymbol{m})$ where $\mathbf{C}_d$ is a covariance operator describing the error ellipsoid within which the data should be fitted (Tarantola 1987) and $\dagger$ denotes the transpose. A similar expression holds for $\Delta_{\mathcal{M}}(\boldsymbol{m}, \boldsymbol{m}_0)$, namely $(\boldsymbol{m} - \boldsymbol{m}_0)^\dagger \mathbf{C}_m^{-1} (\boldsymbol{m} - \boldsymbol{m}_0)$ where $\mathbf{C}_m$ is now a covariance operator describing the ellipsoid within which the model should vary around $\boldsymbol{m}_0$. $\mathbf{C}_d$ and $\mathbf{C}_m$ should be *a priori* information coming not from the given problem but from independent sources. It is, however, extremely difficult to put realistic error bars on seismic data and virtually impossible to estimate data correlations. Thus in tomography the current practice has evolved to reduce $\mathbf{C}_d$ to a diagonal and to decide *a posteriori* (from a few inversions) to what level the data should be fitted. A similar practice applies to

$\mathbf{C}_m$ and λ is an *ad hoc* positive constant. The more philosophical questions arising from such a practice are discussed in Scales and Snieder (1997). In the case of an optimization viewpoint, we ask the question: what is the smallest model subject to the constraint that the data are fitted to an acceptable level? This gives a Lagrange multiplier problem of finding $\boldsymbol{m}$ minimizing C_λ, where λ is the Lagrange multiplier, $\Delta_{\mathcal{D}}(\delta\boldsymbol{d}, \mathbf{A}\boldsymbol{m})$ describes the acceptable fit and $\Delta_{\mathcal{M}}(\boldsymbol{m}, \boldsymbol{m}_0)$ represents the model norm measuring the size of the model (Parker 1994). In both cases, we may interpret (8) as compromising between two characteristics of the model: its size and its disagreement with the data. Both should be as small as possible, but since they cannot usually both go to zero, the cost function (8) is introduced. Requiring C_λ to be as small as possible for a given λ yields the solution. λ is a trade-off parameter which balances the two undesirable properties of the model. The choice of an optimum λ, however, is not an easy one. Alternative forms of cost functions are in use which draw upon the resolving power of the data and the error propagation in the process of inverse mapping (Menke 1989).

2.6. Regularization

We mentioned that the initially ill-posed problem results in a multimapping of an element from the data space into the model space. In that sense, the ill-posedness is related to the fact that the spaces $\mathcal{D}$ and $\mathcal{M}$ are locally defined too large (presence of more or less big nullspaces). Regularization can be seen as locally reducing the sizes of the data and model space, or choosing one particular mapping from many possibilities. In the optimization viewpoint, it can be shown that λ of expression (8) is positive and that the stationary solution corresponding to the minimum model norm is unique (Parker 1994).

Regularization enters the cost function, and hence the solution, in implicit or explicit form. Summarizing, implicit regularization comes from the choice of an upper limit L in the expansion equation (6), the choice of the basis functions themselves and, finally, the choice of a norm in equation (8) since the disagreement of the model with the data is measured by a norm and so is the model size, where the size here means the actual size or more generally the roughness. Explicit regularization is made by deciding upon λ, explicit weightings in the chosen norms and a reference model $\boldsymbol{m}_0$. Regularization transforms the mapping into a well-posed problem and gives us the subjective feeling of a unique solution. I would like to emphasize, however, that the tomographic inverse problem has inherently many solutions and that we have chosen one particular one by the introduction of some regularization. It seems to be desirable to move away from implicit regularization towards explicit regularization due to inconsistency effects (Trampert and Lévêque 1990), nonlinear effects (Sambridge 1990) and leakage problems with truncated basis functions (Trampert and Snieder 1996).

2.7. Inverse operator

To date, most global tomography studies adopt a Bayesian viewpoint, and the most probable solution lies at the minimum of (8). It is possible to minimize (8) using only information of the gradient of C_λ leading to the well known methods of conjugate gradient or steepest descent. Second derivative information of the cost function may also be used to infer the nature of the minimum. Remember that our forward theory has been linearized about a reference model. In a similar way, we now linearize the cost function around a model $\boldsymbol{m}_n$, rather than the reference model $\boldsymbol{m}_0$, where $C_\lambda(\boldsymbol{m}_n) = C_\lambda^n$ and using a second-order Taylor expansion

$$C_\lambda = C_\lambda^n + \nabla C_\lambda^n \delta \boldsymbol{m} + \tfrac{1}{2}\delta \boldsymbol{m}^\dagger \nabla\nabla C_\lambda^n \delta \boldsymbol{m} \tag{9}$$

we solve for the discrepancy $\delta \boldsymbol{m}$ which gives a new model estimate

$$\boldsymbol{m}_{n+1} = \boldsymbol{m}_n + \delta \boldsymbol{m} \qquad \text{with } \delta \boldsymbol{m} = -[\nabla\nabla C_\lambda^n]^{-1} \nabla C_\lambda^n \tag{10}$$

where ∇C_λ^n is the gradient vector and $\nabla\nabla C_\lambda^n$ is called the Hessian. The full calculation of the Hessian is quite computer intensive, but various approximations to the Hessian exist and the Newton approximation (e.g. Tarantola 1987) gives the well known algorithm

$$\boldsymbol{m}_{n+1} = \boldsymbol{m}_n + (\mathbf{A}_n^\dagger \mathbf{C}_d^{-1} \mathbf{A}_n + \lambda \mathbf{C}_m^{-1})^{-1} (\mathbf{A}_n^\dagger \mathbf{C}_d^{-1} \delta \boldsymbol{d} - \lambda \mathbf{C}_m^{-1} \boldsymbol{m}_n) \tag{11}$$

where $\mathbf{A}_n$ is built from the Frechét derivatives of g about $\boldsymbol{m}_n$. Algorithm (11) is iterative and allows us to take account of a slight nonlinearity about the reference model which linearized the forward problem. In most practical cases, the algorithm is stopped after the first iteration to avoid recalculating the partial derivatives in a three-dimensional model.

Different choices of cost functions, different norms and weightings, different trade-off parameters and reference models lead to various inverse operators more or less well suited for a particular problem. Most techniques in use in seismic tomography are discussed in Tarantola (1987), Menke (1989) and Parker (1994). The mathematics is most elegant with the use of L_2-norms, and it is most remarkable that in the linear case many different viewpoints (stochastic, optimization or eigenvector analyses) lead to the same final formulae (Menke 1989). The solution is then described by the same numbers, but of course their interpretation will be quite different depending on the particular viewpoint.

3. And beyond, or what is needed for interpretation

We argued above that global seismic tomography has many multidisciplinary applications. Most interpretations take its results literally, without questioning the 'goodness' of the obtained solution. However, what is the 'goodness' of a solution? Two notions, related to the inherent nonuniqueness of the problem and data uncertainties, are of importance: error analysis and resolution. If there are many possible solutions which fit the data within a certain criterion, error analysis should provide the means of assessing the spread of these solutions and resolution analysis should tell us what can be said about the real Earth. We will show some examples of how interpretation can be misleading by leaving the 'goodness' factor out of the discussion and propose that global optimization techniques are probably best suited for a more rigorous analysis. Another important question, but hardly addressed, is to know how approximations in the forward theory (by linearizing the problem for instance) can affect the estimated model. Here two possible viewpoints may be adopted: include errors in the forward theory in the discussion of model errors, or adopt a more sophisticated forward theory. The latter case almost always involves nonlinearity and most analysing tools are still lacking. In the following, we will examine some unresolved issues in global tomography. We will split this discussion into the linear or linearized problem and the nonlinear case, mainly for convenience, as much, of course, is known in the former case, but statements are more uncertain in the latter case, due to a lack of theoretical background.

3.1. Linear case

3.1.1. Model errors. Modelling of convection based on seismic tomography needs to convert seismic anomalies into thermal anomalies. Hot anomalies represent up-flowing material and cold material is sinking. Basic thermodynamics (and some assumptions) allow us to derive a linear relationship between velocity and thermal anomalies. The sign of the seismic anomalies is thus of primary importance to inferences on convection. In most parts of the lower mantle, seismic anomalies are quite small (rms amplitude lower than 0.5%). Realistic error bars would be important to geodynamicists to help them decide whether the signs of the amplitudes are well constrained. Model uncertainties are usually given under the assumption of Gaussian statistics and expressed by a form of covariance matrix describing the error bars associated with individual model parameters and correlations between the different model parameters. The final model uncertainties are due to imperfect resolution and propagation of data errors. If the prior information on model and data covariances is Gaussian distributed, which is usually assumed in practice, the posterior model covariance is also Gaussian distributed, and it can then be shown (Tarantola 1987) that it is given by the local curvature of the cost function. If the cost function happens to have a very wide valley in which we are picking one particular solution (remember that regularization gives this particular solution), or the underlying statistics are not Gaussian, the local curvature is in general an underestimation of the width of the valley. Error estimation based on Gaussian statistics may well be too optimistic. Similar concerns have been raised for a special case of delay-time tomography by Pulliam and Stark (1994). An important question is then how to use alternative methods of looking at errors. One answer lies in global optimization techniques (Sen and Stoffa 1995). It is out of the question that, for a realistic tomography problem, one can ever assess the whole model space as the number of possible models grows exponentially with the number of model parameters. However, we might be able, with existing computational power, to explore the cost function with Metropolis-type algorithms around the local minimum we found with our linearized inversion. The general ideas of such techniques are well described in Sen and Stoffa (1995).

3.1.2. Resolution. Another central question is: what can be said about the real Earth from our models? Consider the two vertical sections for two different models (figure 2). In the lower mantle, both models are extremely similar, but model S16RLBM (Woodhouse and Trampert 1996) clearly shows more detail than model S12WM13 of Su *et al* (1994) in the upper mantle. Continental roots and to some extent mid-oceanic ridges appear to be much shallower features in model S16RLBM. The point is that discussing the vertical or lateral extent of the features in any model does not make much sense without evoking the corresponding resolution. The finite sampling of the Earth leads to the intuitive understanding that data are able to discern the gross features of the real Earth, but features smaller than a certain characteristic length remain undetected. This critical length is called resolution length. Resolution acts as a linear filter through which the true Earth is seen, true Earth meaning here the Earth which can be modelled with the forward theory. If the resolution operator is the identity operator, the model is perfectly resolved. The most natural way of introducing the concept of resolution is due to Backus and Gilbert (1968). They showed how to construct the inverse operator by assuming a certain shape and width of the resolution filter, or averaging kernel as they called it. Being mathematically a beautiful theory, but practically difficult to implement in the presence of data errors, resulted in their method not finding many applications in seismic tomography. Many different approaches have since been adopted regarding the resolution operator, ranging from ignoring its usefulness, via checkerboard tests, complicated synthetic tests simulating reality to formal calculations of the operator itself. In a beautiful little research note, Lévêque *et al* (1993) showed that, unfortunately, the only way of estimating the resolution is by explicitly calculating it. It might also be useful to remark that Backus and Gilbert's averaging kernel refers to the physical model (called here $\delta v(r)$), while the resolution operator for most tomographic inversion refers to the coefficients in equation (6). It is of course trivial to switch from one model space to the other, if the basis functions are explicitly known, but there still seems to be some confusion in the literature. A typical tomographic problem tries to find at least several thousands of unknowns. The fundamental problem of resolution is not to calculate the operator (all that is needed is computer time and some patience), but its visualization for interpretation. Backus and Gilbert's philosophy taken more liberally might be the answer to that problem. We are free to choose the parametrization of our inverse problem, keeping in mind the possible consistency problems related to implicit regularization. If we could find an optimum parametrization such that the corresponding resolution is as close to the identity operator as possible, the resolution would be implicitly contained in the parametrization. This implies closely spaced basis functions in well-resolved areas and widely spaced basis functions in badly resolved areas. Recently, Curtis and Snieder (1997) showed how global optimization techniques may be used for such a purpose and, alternatively, Sambridge and Gudmundsson (1998) proposed general algorithms for flexible irregular cell parametrization of ray-based tomography problems. To complicate matters, resolution is only half the story. Trampert and Snieder (1996) pointed out that truncating the expansion in equation (6) at an upper limit L, the neglected basis functions may leak into the estimated coefficients, if the sampling of the Earth is not homogeneous. The full resolution equation is then given by

$$\boldsymbol{m}_L^{\mathrm{est}} = \mathbf{R}\boldsymbol{m}_L^{\mathrm{true}} + \mathbf{B}\boldsymbol{m}_\infty^{\mathrm{true}} \tag{12}$$

where $\mathbf{R}$ is the classical resolution operator discussed above and $\mathbf{B}$ is a bias operator which describes how the neglected basis functions leak into the L-estimated coefficients. This bias is not negligible for realistic tomography problems, but so far it has not explicitly been taken into account. A close analysis of $\mathbf{B}$ shows that leakage is strongest for coefficients close to the truncation level, and hence second-derivative model smoothing very efficiently counteracts the effects of the bias operator. In the absence of any sound prior information regarding the solution, it is thus desirable to implement second-derivative model smoothing, which is far easier to use numerically than formal leakage correction.

3.1.3. Errors in the forward theory. The notion of 'true' Earth is intimately related to how the data can be predicted, or the correctness of the forward theory. It is then important to know the effects of theoretical errors upon the estimated solution. Tarantola (1987) and Jackson (1994) addressed this problem under the assumption that the errors in the forward modelling can be described by a certain statistical distribution. The main restrictions in the forward modelling of seismic tomography are the stationarity principles. The extent to which nonray-theoretical effects may corrupt the models constructed so far can only be determined by extensive numerical simulations using more advanced algorithms for three-dimensional ray tracing and the calculation of synthetic seismograms in a three-dimensional Earth. So far, no quantitative assessments of these effects have been performed.

3.2. Nonlinear case

Nonlinear problems are far more difficult to analyse. Part of the nonlinearity of the seismic tomography problem comes from the fact that the paths which elastic waves travel along depend upon the structure they travel through. Under some assumptions, stationarity principles may be used to linearize the problem. However, if the underlying structure is too heterogeneous or the data we want to use are not stationary quantities (e.g. inverting amplitude for quality factor), the only viable approach seems to be to explore the total model space by forward modelling and testing against the data. Such techniques can be regrouped under the name of global optimization and a detailed account is given by Sen and Stoffa (1995). Unfortunately, the number of possible models grows exponentially with the number of model parameters which makes such an approach virtually impossible for a realistic seismic tomography problem.

In most practical cases, nonlinearity has been treated locally, so that algorithms such as (11) may be used. Alternatively, Snieder (1991) showed how to extend Backus and Gilbert's (1968) theory to the nonlinear case. He again used a local approach based on perturbation theory. A more promising direction is to look for systematic transformations, which reformulate the nonlinear inverse problem into a linear one. For instance, Woodhouse (1978) has shown how the velocity as a function of depth can be obtained by a linear inversion of travel times observed as a function of distance between a source and a receiver choosing an appropriate parametrization. Classically, this problem has been solved by reducing the travel distance in terms of a velocity–depth function to the form of Abel's integral equation, for which a nonlinear exact inverse exists under certain assumptions. In another approach of transforming the original problem, Grünbaum (1992) and Grünbaum and Zubelli (1992) developed a technique, called diffuse tomography, where the nonlinear forward problem can be solved by a finite set of linear systems of equations. To my knowledge, this has not found any applications in geophysics yet, but there are great similarities with the seismic tomography problem involving multiple scattering. Recently, Vasco (1997) presented a general approach which transformed nonlinear inverse problems into linear ones based on the theory of continuous groups. Clearly, at this stage not much more can be said about fully nonlinear tomography, but it is a clear challenge for future work.

4. Concluding remarks

In the last 20 years, global seismic tomography has made tremendous progress in the mapping of three-dimensional elastic wave velocity fields. Many robust pieces of information have emerged which stimulated a multidisciplinary discussion within the Earth sciences. This information is becoming increasingly important in understanding the internal machinery of our planet. We have now reached a point where we have to look more closely at the effects of approximations in our modelling. Error and resolution analyses have to be made in a more systematic and realistic way to promote a more robust interpretation of the results. The data quality is of course an important factor in the advance of seismic tomography, but assuming a steady increase of data quality and coverage, the fully nonlinear treatment of global tomography is the most promising direction for substantial improvement in our understanding of the Earth's deep interior. In many cases, this involves complications in the mathematics of the problem, and it is hoped that scientists from outside the Earth sciences, with similar problems, may shed a new light on our approaches.

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