Path-average kernels for long wavelength traveltime tomography

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SUMMARY

Although much effort goes into improving the resolution of tomographic models, investigating their quality has only just started. Probabilistic tomography provides a framework for the quantitative assessment of uncertainties of long-wavelength tomographic models. So far, this technique has been used to invert maps of surface wave phase velocities and normal-mode splitting functions. Including body waves would substantially increase the depth resolution in the lowermost mantle. In surface wave tomography, the construction of phase velocity maps and splitting functions is a well-defined inverse problem, and the depth inversion is less well constrained but characterized by a small number of dimensions suitable for a Monte Carlo search. Traveltime tomography is mostly based on ray theory that covers the 3-D Earth, thus the dimension of the inverse problem is too large for a Monte Carlo search. The ray-mode duality suggests to apply the path-average approximation to body wave traveltimes. In this way the measured traveltime residual as a function of ray parameter can be inverted using path-average kernels, which depend on depth only, similar to surface wave tomography.

We investigate the validity of the path-average approximation for delay times in both the forward and the inverse problem using the velocity model S20RTS as well as random models. We numerically illustrate the precision of such kernels compared with ray-theoretic and finite-frequency ones. We further invert traveltime residuals, calculated from Fermat rays, using the path-average kernels. We find that the agreement between classical ray theory and path-average theory is good for long wavelength structures. We suggest that for mapping long wavelength structures, body waves can be inverted in two steps, similar to surface waves, where the ray parameter and the vertical traveltime play the role of frequency and phase velocity, respectively.

Key words: Body waves; Seismic tomography; Wave propagation.

1 INTRODUCTION

Seismic tomography is the most powerful probe into the Earth's deep interior. Seismic waves from large earthquakes travel through the Earth or along the surface, and their arrival times and shapes contain information on the medium they travelled through. Tomography inversions consists of working out the Earth's 3-D (an)elastic structure (generally expressed as a perturbation from a reference model) from large quantities of arrival times, body or surface wave forms and free oscillation data, and has been a very active field of research since the first systematic efforts in the early 1980s. Over the years, considerable progress has been made and many reviews have been written on the subject (e.g. Dziewonski & Woodhouse 1987; Woodhouse & Dziewonski 1989; Masters 1989; Romanowicz 1991; Montagner 1994; Masters & Shearer 1995; Ritzwoller & Lavely 1995; Dziewonski 1996; Masters et al. 2000; Fukao et al. 2001; Romanowicz 2003; Trampert & van der Hilst 2005). It is encouraging to see that a great amount of overlapping information is emerging from different tomographic studies using different data and/or different techniques.

The imaging of the 3-D velocity structures is of course only a first step towards fully understanding their physical causes. Indeed, seismic tomography maps the current thermodynamic and compositional state of heterogeneity and thus imposes strong geometric constraints on possible models of mantle convection. The thickness of continental roots (Deschamps et al. 2002), the depth extent of the mid-ocean ridges and the change of lithospheric velocity versus age (e.g. Su et al. 1992; Zhang & Tanimoto 1992) give important clues on the formation and evolution of continental and oceanic lithosphere. It has long been recognized that there is a correlation between the geoid and seismic models, and therefore judicious integration of both types of information (gravity and seismological) should give access to robust 3-D density variations within the Earth (e.g. Ishii & Tromp 1999). These few examples illustrate that seismic tomography is central in a strong interdisciplinary effort aimed at understanding the structure and the evolution of the Earth's interior.

Although much effort has been made to improve the lateral and depth resolution of the models, investigating their quality has only just started. Model appraisal, however, is essential if we want to advance from a predominantly qualitative to a more quantitative interpretation of seismic tomography (Trampert & van der Hilst 2005). It is well known that the tomographic problem is an illposed and ill-conditioned inverse problem (e.g. Trampert 1998). The ill-ness is reduced by imposing some form of regularization. This improves the conditioning and keeps the data error propagation under control. Ill-posed means that the inferred model is not unique (model nullspace), some data cannot be associated to any model (data nullspace), or the model does not depend continuously on the data (Bertero & Boccacci 1998). The last condition can lead to multiple minima in the cost function, even for a linear inverse problem and a standard regularization could therefore find a solution in a local minimum which might not be the most likely. Beghein et al. (2002), Resovsky & Trampert (2003) identified two such cases while studying anisotropy in the inner core and density in the mantle, respectively. The presence of nullspaces is more generally acknowledged, and it is well known that the model nullspace leads to large valleys in the data misfit function, which a least-squares method seeks to minimize. In this case, regularization is needed to eliminate the valley. This makes the cost function quadratic and the solution unique. Regularization can thus be seen by picking one particular solution from the original valley in the data misfit function. In the Bayesian approach (e.g. Tarantola 1987), the regularization consists of statistical prior information, which is obtained by other physical experiments. If the regularization is just chosen by ad hoc mathematical arguments, as in most cases, the solution loses connection to the real world and formal uncertainty analysis is meaningless. In the absence of realistic prior information, the valley should be properly explored by a full model search technique. In this case, a family of solutions is obtained, and it emerged that often their statistics have useful and interpretable properties due to sufficient data constraints (Shapiro & Ritzwoller 2002; Resovsky & Trampert 2003). We refer to the statistical constraints from such a family of solutions as 'probabilistic tomography' (Resovsky & Trampert 2003; Trampert et al. 2004).

Based on a forward sampling technique, probabilistic tomography associates a probability to each random model based on some definition of misfit. When sufficient models have been drawn, probability density functions are estimated for each model parameter. This probability density function is all the information that can ever be gained from the data on that particular parameter and can therefore be seen as a compact representation of the given data themselves. The curse of dimensionality quickly limits the size of the problem, which can be solved with a forward sampling technique to a few tens of unknowns (e.g. Curtis & Lomax 2001). So far this technique has been used to invert surface wave phase velocity maps and normal-mode splitting functions (e.g. Beghein et al. 2002; Resovsky & Trampert 2003). The construction of phase velocity maps and splitting functions from the seismic measurements is a relatively well-defined linear inverse problem. Their local depth inversion, however, is less well constrained and the biggest source of non-uniqueness in the final models, but a problem which can be represented with a limited number of unknowns. The latter part is therefore ideally suited for probabilistic tomography. So far, body wave data have not been included into probabilistic tomography, although they would considerably increase depth resolution. Body wave traveltime residuals are generally analysed using ray theory, which covers the 3-D Earth. It is therefore not straightforward to decompose the problem into a mapping of body wave residuals with a function on the sphere which can then locally be inverted for depth. Using rays means parametrizing the whole Earth, which makes the problem too big for probabilistic tomography.

Evoking the ray-mode duality, it is possible to apply the pathaverage approximation to body wave traveltimes in a spherically symmetric earth model. In the following section, we show that the measured traveltime residual as a function of ray parameter can, to lowest order, be inverted using a simplified kernel which depends only on depth. In Section 3, we show comparisons of traveltime residuals calculated using Fermat rays, finite-frequency kernels and the path-average approximation using 3-D earth models with varying complexity. In Section 4, we then illustrate that the path-average approximation, commonly used for surface waves and normal-mode splitting, gives good inversion results for long wavelength structures when applied to body wave residuals.

2 SIMPLE KERNELS FROM RAY-MODE DUALITY

The path-average approximation has successfully been used in many long-period waveform inversions (e.g. Woodhouse & Dziewonski 1984, 1986; Tanimoto 1988; Su et al. 1994; Kustowski et al. 2008). The theory shows that for minor arc phases, the phase adjustment to each mode in the seismogram depends on the average phase velocity perturbation between source and receiver and therefore only on the horizontally averaged structure. This is, of course, a very good approximation for surface waves but not necessarily for body waves. Li & Romanowicz (1995) proposed an extension to the theory, which involves cross-branch coupling, and investigated its limitations for the calculation of body waveforms. The good agreement of the long wavelength structure between many tomographic models shows that the path-average approximation nevertheless provides meaningful results. It can be shown (e.g. Dahlen & Tromp 1998) that the constraints on the Earth's internal structure from high-frequency normal modes and body wave traveltimes are the same to the lowest order. Secondary data popularly derived from body waveforms are traveltime residuals. We therefore suggest to employ the path-average approximation for the description of body wave traveltime residuals rather than full waveforms.

In a spherically symmetric earth model, the normal-modes of the Earth can be seen as the constructive interference of body waves with the same ray parameter. Brune (1964, 1966) was the first to analyse body wave arrival times in terms of equivalent normal-mode frequencies using simple arguments of constructive interference. For large angular orders l, the normal-mode frequencies are asymptotically equivalent to the inverse intercept time of a ray with ray parameter p:

$$\omega\tau(p) = 2\pi(n+\alpha),\tag{1}$$

where ω is the angular frequency of the mode, *n* its overtone number and α is a fraction which depends on the body wave considered (Zhao & Dahlen 1993). The ray-theoretic intercept time is defined by:

$$\tau(p) = 2 \int_{r(\text{bot})}^{a} \left(\frac{1}{v(r)^2} - \frac{p^2}{r^2} \right)^{1/2} \mathrm{d}r,\tag{2}$$

where v(r) is the wave speed of either the *P* or the *S* wave under consideration and the integration runs from the turning radius of the ray to the Earth's surface *a*. Together with Jean's relation (1927) this allows to calculate the mode spectrum of a spherically symmetric Earth with remarkable precision (for high *l*) compared with the direct integration method, which is the standard (see illustrations in Zhao & Dahlen 1993). The ray-mode duality goes much further and by rewriting the constructive interference principle of eq. (1) differently, Zhao & Dahlen (1995a) arrived at asymptotic expressions for the modal eigenfunctions and the corresponding Fréchet kernels (Zhao & Dahlen 1995b). A local eigenfrequency perturbation at constant *l* is related to a local phase velocity perturbation at constant ω (e.g. Dahlen & Tromp 1998) by

$$\frac{\delta\omega_{\text{local}}}{\omega} = \frac{U}{c} \frac{\delta c_{\text{local}}}{c},\tag{3}$$

where $U = \Delta(p)/T(p)$ is the group velocity defined simply as the ratio of the epicentral distance and the traveltime, and $c = p^{-1}$ is the phase velocity of the normal-mode corresponding to a body wave with ray parameter *p* (Dahlen & Tromp 1998). Inserting these in the expressions of Zhao & Dahlen (1995b), the local vertical traveltime perturbations are related to local velocity perturbations by

$$\delta\tau(p,\theta,\phi) = -2\int_{r(\text{bot})}^{a} \frac{1}{v(r)^2} \left(\frac{1}{v(r)^2} - \frac{p^2}{r^2}\right)^{-1/2} \delta\ln v(r,\theta,\phi) \,\mathrm{d}r.$$
(4)

The physical meaning of this expression is that a ray with ray parameter p in a spherically symmetric earth model v(r) experiences a local vertical two-way traveltime perturbation due to a local velocity perturbation at point (r, θ, ϕ) in the Earth (Fig. 1). Note that eq. (4) is simply the differential of (2) assuming a local velocity perturbation for a constant p. The equation is valid for P, PcP, S and ScS waves, which all have one down- and one up-ward leg to their path. For PP and SS waves, the factor of 2 is replaced by a factor of 4, for PPP and SS waves by a factor of 6, etc., corresponding to the number of legs in their path. We assumed here that the source is at the surface, but a generalization to an epicenter at depth is straightforward. These kernels very strongly resemble those of Zhao & Jordan (1998) for the case of along branch coupling which isolates the minor arc sensitivity. The observed traveltime residual $\delta T(p)$



Figure 1. The local two-way vertical traveltime perturbation $\delta \tau$ is obtained by integrating a local velocity perturbation δv at the point (r, θ, ϕ) in a spherically symmetric Earth. The kernel is schematically depicted by bands of varying grey shades (black is the maximum). The total traveltime perturbation is determinated by averaging laterally the local $\delta \tau$. For an anomaly close to the bottom of the ray, depicted in green, the ray-theoretical and path-average residuals are close. For an anomaly close to the surface, the difference will be large.

for this ray is then simply the average $\delta \tau(p, \theta, \phi)$ between source and receiver, which is the body wave equivalent of the path-average approximation for modal frequencies:

$$\delta T(p) = \frac{1}{\Delta} \int_0^\Delta \delta \tau(p, \theta, \phi) \, \mathrm{d}\Delta, \tag{5}$$

where Δ is the epicentral distance. This formulation is equivalent to that of surface wave tomography, where the role of frequency ω is taken by the ray parameter p and the local phase velocity perturbation is replaced by the vertical traveltime perturbation.

So far, we approached the problem from the modal description and used asymptotic expressions for high l or high ω . The same kernels can of course be found by starting from ray theory, the high-frequency approximation to the elastodynamic wave equation. Fermat's principle states that the traveltime of a ray is extremum, minimum or maximum, with respect to nearby possible paths. This means that the traveltime perturbation due to a velocity perturbation is to first order:

$$\delta T(p) = -\int_{\Gamma} \frac{\delta \ln v(r, \theta, \phi)}{v(r)} \,\mathrm{d}\Gamma,\tag{6}$$

where Γ denotes the ray path. If we assume that the velocity perturbation between source and receiver depends on radius only, we can transform the integral over Γ into an integral over radius *r* (e.g. Bullen 1963):

$$\delta T(p) = -2 \int_{r(\text{bot})}^{a} \frac{1}{v(r)^2} \left(\frac{1}{v(r)^2} - \frac{p^2}{r^2}\right)^{-1/2} \delta \ln v(r) \,\mathrm{d}r.$$
(7)

It is interesting to note that this is how Bullen constructed his 1-D velocity models. In the Earth, the velocity perturbation of course changes laterally as well, but if we are content to retrieve a lateral average of this perturbation between source and receiver, we arrive again at eq. (5).

To address the validity of the path integral approximation, we need to estimate by how much the velocity can change laterally for (5) to remain valid. By considering a small segment of a ray, it is easy to see that the approximation holds locally when

$$\frac{\delta \ln v_{\perp}}{\delta \ln v_{r}} \ll \tan i, \tag{8}$$

where *i* is the angle of incidence. This equation shows that the path integral approximation is acceptable when the horizontal changes in velocity are small in parts, where the ray travels vertically. Near the bottom of the ray, where the propagation is close to horizontal, no limitations are needed. The local condition (8) cannot be generalized to a global constraint, but it is clear that the precision of eq. (5) will depend on the distance and the wavelength of the structural changes. In the following, we will numerically illustrate the precision of the path-average kernels using random media and the velocity model S20RTS (Ritsema *et al.* 1999).

3 THE FORWARD PRECISION

S20RTS (Ritsema *et al.* 1999) is a 3-D tomographic velocity model expressed as perturbations from PREM (Dziewonski & Anderson 1981). The parametrization consists of a spherical harmonic expansion (up to degree 20) of the lateral perturbations and a spline expansion (21 splines) of the vertical perturbations between the Moho and the core–mantle boundary. We traced rays in PREM and calculated ray-theoretic traveltime perturbations in S20RTS using eq. (6). We then compared those with path-average predictions



Figure 2. Scatter-plots comparing ray-theoretic and path-average approximations to traveltime perturbations for *S*, *SS*, *ScS*, *SSS* phases and a spherical harmonic cut-off 5. The red line is the fitted line obtained by linear regression, and the green line has the ideal slope 1. Each graph displays results for 1000 random source–receiver paths.

using eq. (5). The complexity of the model is varied by considering different spherical harmonic cut-offs. We generated a random distribution of source–receiver paths. We fixed a certain epicentral distance and randomly generated epicenter coordinates, azimuths (between 0° and 360°) and source depth (between 0 and 700 km) and calculated the corresponding station coordinates. For each seismic phase, we generated 1000 paths and plotted the ray-theoretic versus the path-average traveltime perturbations.

Figs 2 and 3 show scatter-plots for S, SS, SSS and ScS waves for different spherical harmonic degrees and epicentral distances. The plots exhibit a clear linear trend and in the ideal case, all points should align on the green line with slope 1. Instead, they scatter around the fitted red line obtained by linear regression. The fitted line is often close to the ideal line, but we see that the deviations from the green line increase with spherical harmonic degree and epicentral distance as expected. To quantify these deviations, we calculated the linear correlation coefficient (Fig. 4), which is a measure of the probability that a linear relationship exists between the two residuals. Fig. 4 depicts the correlation coefficient as a function of epicentral distance and spherical harmonic degree. In general, it is higher than 0.7 and decreases with increasing spherical harmonic degree and epicentral distance. Plots for the SSS phase show a very high correlation. This is due to the nature of SSS waves. With increasing bounce points, properties of body waves are closer to properties of surface waves. Since the path-average

approximation works very well for surface waves, it is also a good approximation for SS and, in particular, SSS phases.

Another important measure is the deviation from ray theory. We defined the relative uncertainty as the ratio between the standard deviation from the line with slope 1 and the quadratic mean of points described by the green line. We chose to show the relative uncertainty, because it is independent on the amplitude of the model. In Fig. 5, which exhibits the relative uncertainties in S20RTS, there is a slight dependence on distance and, hence, on ray parameter. In an inverse procedure, this theoretical uncertainty would simply add to the measured data uncertainties (e.g. Tarantola 1987).

Ray theory, a high frequency approximation, is only valid if the structure changes little on the scale of the wavelength and/or the Fresnel zone (Wang & Dahlen 1995). In practice, the waves are of finite frequency and scattering effects might be important. Dahlen *et al.* (2000) formulated an efficient theory for calculating such finite-frequency sensitivity kernels for body-wave arrival times. These kernels have banana–doughnut shapes for direct phases and have a strong and narrow sensitivity close to the source and receiver and weaker and broader sensitivity near the turning point (e.g. Hung *et al.* 2000). Our simplified kernels differ most near the source and receiver, and therefore it is interesting to compare finite-frequency kernel predictions with our predictions. We used the code 'raydyn-trace.f' (Tian *et al.* 2007a), together with the paraxial approximation to calculate the finite-frequency kernels assuming a dominant



Figure 3. Same as Fig. 2, but up to spherical harmonic degree 20.



Figure 4. Linear correlation coefficient as a function of epicentral distance and spherical harmonic degree for S, SS, ScS, SSS phases.

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Figure 5. Relative uncertainty between ray- and path-average theory in S20RTS as a function of epicentral distance and spherical harmonic degree for *S*, *SS*, *ScS*, *SSS* phases.



Figure 6. Linear correlation coefficient (top panels) and relative uncertainty (bottom panels) as a function of epicentral distance, and spherical harmonic degree for *S* and *ScS* phases comparing the banana–doughnut and the path-average approximation.

period of $\tau = 20$ s. We then integrated the kernel multiplied with the velocity perturbation numerically around the first few Fresnel zones.

The scatter-plots of banana–doughnut versus path-average traveltime pertubations for S20RTS are very similar to those in Figs 2 and 3, indicating that finite-frequency effects are small, which is to be expected for long wavelength models. In Fig. 6, we show the linear correlation coefficient (top panel) and the relative uncertainty (bottom panel) as functions of epicentral distance and harmonic degree. They are very close to those in Figs 4 and 5. The paraxial approximation breaks down for waves producing caustics (Tian *et al.* 2007b). Our simple integration scheme can therefore not be applied to *SS* and *SSS* phases, restricting our banana-doughnut comparison to *S* and *ScS* waves only.

Some of the features in the previous figures are due to the nature of S20RTS, which has a varying spectrum as a function of depth.



Figure 8. Same as Fig. 5, but using a random model.

Inspired by Baig *et al.* (2003), we investigated the behaviour of our path-average kernels generating random models characterized by flat spherical harmonic power spectra. Again, we compared the ray-theoretic and the path-average traveltime residuals using now random velocity models for different spherical harmonic cut-offs and epicentral distances. Fig. 7 (Fig. 8) shows the correlation coefficient (the relative error) as a function of epicentral distance and spherical harmonic degree. Because we drew random spherical harmonic coefficients between (-1, 1), the amplitude of the traveltime

residuals has values between ~ 100 and ~ 3000 s, compared with less than 20 s for S20RTS. The relative error, however, is comparable to those exhibited for S20RTS. This shows that relative uncertainty doesn't depend on the amplitude of the model.

Comparing Figs 4 and 7 (5 and 8), the trend of the correlation (relative uncertainty) is the same for *SS*- and *SSS*-phases, but it is different for *S*- and *ScS*-waves. For the model S20RTS, the relative uncertainty decreases at long distances, whereas it increases with increasing distance for random media. This effect is most

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evident for the *S*-phase. The explanation is that S20RTS is dominated by harmonic degree 2 in the lowermost mantle. The effects of the cut-off degree are clearer in the random models, which give uncertainties due to the path-average approximation, unbiased by their power spectrum.

4 THE INVERSE CASE

In the previous section, we saw that the path-average approximation provides a good estimation of traveltimes in long-wavelength earth models. We now present the results of a tomographic inversion of body-wave traveltime residuals for realistic source–receiver paths. We used 73 394 *S*-, 60 114 *SS*- and 3020 *ScS*-waves (Ritsema, personal communication, 2008). To construct velocity perturbation maps, we calculated traveltime residuals in S20RTS using Fermat rays and then inverted for d ln v_s using simplified kernels from the path-average approximation (eq. 5).

S20RTS is parametrized laterally with spherical harmonic functions up to degree 20 and 21 spline functions for depth. The number of unknows therefore is 9261. The inversion is performed adopting a damped least-squares method, which introduces a trade-off parameter γ to find a solution by minimizing a cost function



Figure 9. Shear velocity perturbation maps of model S20RTS on the left-hand side, from the inversion using classical ray theory in the centre and the path-average approach on the right-hand side at different depths. Ray-theoretic traveltime data were calculated up to degree 20. The perturbations are given in per cent with respect to PREM. Yellow circles are hotspots and yellow lines represent plate boundaries.

 $S = ||\vec{d} - \hat{A}\vec{m}||^2 - \gamma ||\vec{m}||^2$, where \vec{d} is the data vector, \vec{m} the model vector and \hat{A} the partial derivative matrix. The trade-off (or damping) parameter compromises between minimizing the data misfit and the size of the model.

Maps of relative deviation from the average shear wave velocity (PREM) at seven depths are shown in Fig. 9. For each depth, we reproduce model S20RTS on the left-hand side, results from the inversion using classical ray theory in the centre and the path-average theory on the right-hand side. Blue regions denote faster than average and red regions denote slower than average velocity. For shallow depths (between 150 and 500 km), there are clear discrepancies in amplitude between S20RTS, and the other two results due to data coverage used in our test. Indeed, body waves are more suitable to study the deeper interior of the Earth and thus to detect heterogeneity in the lower mantle; whereas surface waves resolve better smallscale structure in the upper mantle. Since model S20RTS combines body waves, surface waves and normal-modes, it provides better constraints in the shallow part of the Earth's mantle. Increasing the depth, the agreement in terms of amplitude between maps improves considerably, and in particular, in the lowermost mantle, the negative velocity anomalies beneath the Pacific and Africa from the three maps show a similar amplitude. Comparing S20RTS to the middle column in Fig. 9 is like a checkerboard test and gives an idea on the resolution power of the data for the chosen damping; the comparison between the middle and right-hand side column shows the difference between ray- and path-average theory.

The differences and similitaries between maps in Fig. 9 appear more clearly by directly computing their correlations, as well as comparing their amplitudes.

Fig. 10 shows the correlation between S20RTS and the pathaverage model on the left-hand side and between classical ray theory, and our approach on the right-hand side as a function of the depth for different harmonic degrees. We note that in the upper mantle, there are less differences in the trends of the correlation with increasing spherical harmonic degree. Instead, in the mid- and lower mantle, the long wavelength features correlate much better. In the lower mantle, the lowest correlation is around 1500 km corresponding to a distance $\sim 60^{\circ}$. S20RTS is characterized by small amplitudes at that depth, and the error due to the path-average approximation becomes comparable to the data value itself. Of course, this affects the correlation between velocity perturbation maps as well. We do not think that this drop in correlation is a problem for imaging the real Earth, because S20RTS seems quite heavily damped compared with other models using similar data (Trampert & Spetzler 2006) but is a reminder that it remains difficult to extract signal from data below the noise level.

In Fig. 11, we show the amplitude ratios between S20RTS and the inverted maps as function of depth and maximum harmonic degree. The trend depends little on harmonic degree. The amplitude ratio between S20RTS and the path-average model is more than 7 in the uppermost mantle, then declines rapidly around 1.0. The amplitude ratio between maps from ray theory and path-average theory decreases from 2.5 in the upper mantle to 1 in the lower mantle. The reason is that the ray-theory maps and path-average maps are constructed using only body waves, which do not provide a good coverage in the upper mantle.

To show that body waves and surface waves can be treated using the same formalism, we applied a two-step inversion to the Fermat traveltime residuals used above. We expanded traveltime data for ray parameter, where we had enough data coverage in spherical harmonics up to degree 8 and then inverted the maps of $\delta T(p)$ locally for depth. In practice, rays with constant ray parameter are



Figure 10. Correlation as a function of the depth for different maximum harmonic degrees between the S20RTS and the path-average model on the left-hand side and between the classical ray theory and path-average maps on the right-hand side.



Figure 11. Same as Fig. 10, but for the amplitude ratio as a function of the depth for different maximum harmonic degrees.

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Figure 12. Sensitivity kernels for the path-average theory as function of the depth and ray parameter for selected *S*- (left-hand bottom panel) and *SS*-phases (right-hand bottom panel).

difficult to find (except for diffracted phases). We therefore construct maps of $\delta T(p)$ using rays in a narrow range. Sufficient rays are found for *S* waves with ray parameter p = 479 and 499.5 ± 0.5 s deg⁻¹ and for *SS* waves bottoming between 970 and 1100 km with a range of *p* equal to 5 s deg⁻¹. Fig. 12 shows the path-average sensitivity kernels corresponding to the selected rays. Fig. 13 exhibits *S*-wave velocity perturbation maps obtained from this two-step inversion. The agreement with the model S20RTS (left-hand column of Fig. 9) in terms of pattern and amplitude is quite good only at a depth of 1000 km and in the lowermost mantle. The reason is of course that the maps are obtained with only *S* and *SS* phases, which bottom at depth ~2890 and ~1000 km, respectively.

5 DISCUSSION AND CONCLUSION

In the asymptotic limit, it is possible to express a duality between high-frequency normal-modes and propagating body waves. This ray-mode duality suggests to apply the path-average approximation, which is typically used for normal-modes and surface waves, to body wave traveltimes. We expressed the traveltime residual between source and receiver as the lateral average of the local vertical twoway traveltime perturbation. In this way, surface and body waves can be treated using the same formalism, where the frequency ω corresponds to the ray parameter p and the local phase velocity perturbation is comparable to the vertical traveltime perturbation. It is therefore possible, in theory, to build traveltime residual maps as functions of the ray parameter comparable to phase velocity maps as functions of frequency and invert those locally for depth.

The aim of this paper was to test the validity of such path-average kernels for both the forward and the inverse problem. Although we only showed results for *S, SS, SSS, ScS* phases, we tested that the same approach works for *P, PP, PPP, PcP* phases.

In body wave tomography, it is important to take into account the errors due to the source mislocations. Two different approaches have been proposed to deal with this. (1) The relocation procedure and the tomographic inversion can be performed simultaneously (Bijwaard & Spakman 2000). (2) Earthquake relocation parameters are estimated by fitting observed traveltimes to predictions in a smooth degree-12 velocity model (Ritsema *et al.* 2004). A relocation error can then be estimated and simply subtracted from the measured traveltime residuals before the tomographic inversion. This latter procedure is most easily incorporated in a two-step path-average approach.



Figure 13. Shear velocity perturbation maps at different depths from the two-step inversion. The first step consists of a spherical harmonic expansion of the measured traveltime residuals up to degree 8, and the second one is their depth inversion using the kernels in Fig. 12. The perturbations are given in per cent with respect to PREM. The maps show hotspots (yellow circles) as well as plate boundaries (yellow line).

Using the velocity model S20RTS (Ritsema *et al.* 1999) and random media, we compared the ray-theoretic delay time and the banana-doughnut traveltime residual with the traveltime perturbation, using simple kernels from the path-average approximation for different distances and spherical harmonic cut-offs. For longwavelength structure, the agreement is good, and it improves further for low epicentral distances. We estimated a relative uncertainty that is independent of the amplitude of the model and depends slightly on the distance and, thus, on the ray parameter. It is not necessary to establish strict bounds of validity for the path-average approach, but relative uncertainties can properly be included in the construction of the tomographic models.

To test the effectiveness of the path-average approximation for body waves in tomographic inversions, we used traveltime residuals, calculated from Fermat theory, to image the velocity perturbation using both classical ray theory and the theory presented here. For long wavelength models, the results are comparable in the lower mantle. Our tests show that a two-steps approach is also possible, but data coverage is such that it cannot be used for body wave tomography alone. However, two-way traveltime maps can be added to surface-wave phase velocity maps and normal-mode splitting functions. This allows to construct mantle models using the same mathematical formalism for body and surface waves.

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REFERENCES

- Baig, A.M., Dahlen, F.A. & Hung, S.-H., 2003. Traveltimes of waves in three-dimensional random media, *Geophys. J. Int.*, 153, 467–482.
- Beghein, C., Resovsky, J.S. & Trampert, J., 2002. P and S tomography using normal mode and surface wave data with a neighbourhood algorithm, *Geophys. J. Int.*, **149**, 646–658.
- Bertero, M. & Boccacci, P., 1998. Introduction to Inverse Problem in Imaging, IOP Publishing, Bristol.
- Bijwaard, H. & Spakman, W., 2000. Non-linear global P-wave tomography by iterated linearized inversion, *Geophys. J. Int.*, 141, 71–82.
- Brune, J.N., 1964. Travel-times, body waves and normal modes of the Earth, *Bull. seism. Soc. Am.*, **54**, 1315–1321.
- Brune, J.N., 1966. P and S travel-times and spheroidal normal modes of a homogeneous sphere, J geophys. Res., 71, 2959–2965.
- Bullen, K., 1963. Introduction to the Theory of Seismology, Cambridge University press, London.
- Curtis, A. & Lomax, A., 2001. Prior information, sampling distributions and the curse of dimensionality, *Geophysics*, 66, 372–378.
- Dahlen, F.A. & Tromp, J., 1998. Theoretical Global Seismology, Princeton University Press, NJ.
- Dahlen, F.A., Hung, S.-H. & Nolet, G., 2000. Frechet kernels for finite frequency traveltimes, I: theory, *Geophys. J. Int.*, 141, 157–174.
- Deschamps, F., Trampert, J. & Snieder, R., 2002. Anomalies of temperature and iron in the uppermost mantle inferred from gravity data and tomographic models, *Phys. Earth planet. Int.*, **129**, 245–264.
- Dziewonski, A.M., 1996. Earth's mantle in three dimensions, in *Seismic Modelling of Earth Structure*, Vol. 25, pp. 507–572, Instituto Nazionale di Geofisica, Rome.
- Dziewonski, A.M. & Anderson, D.L., 1981. Preliminary reference earth model, *Phys. Earth planet. Int.*, 25, 297–356.
- Dziewonski, A.M. & Woodhouse, J.H., 1987. Global images of the Earth's interior, *Science*, 236, 37–48.
- Fukao, Y., Widiyantoro, S. & Obayashi, M., 2001. Stagnat slabs in the upper and lower mantle transition region, *Rev. Geophys.*, 39, 291–323.
- Hung, S.-H., Dahlen, F.A. & Nolet, G., 2000. Frechet kernels for finite frequency traveltimes, II: examples, *Geophys. J. Int.*, 141, 175–203.
- Ishii, M. & Tromp, J., 1999. Normal-mode and free-air gravity constraints on lateral variations in velocity and density of the Earth's mantle, *Science*, 285, 1231–1236.

- Kustowski, B., Ekstrom, G. & Dziewonski, A.M., 2008. Anisotropic shear wave velocity structure of the Earth's mantle: a global model, *J. geophys. Res.*, **113**, B06306, doi:10.1029/2007JB005169.
- Li, X.-D. & Romanowicz, B., 1995. Comparison of global waveform inversions with and without considering cross-branch modal coupling, *Geophys. J. Int.*, **121**, 695–709.
- Masters, G., 1989. Low frequency seismology and the three-dimensional structure of the Earth, *Phil. Trans. R. Soc. Lond. A*, **328**, 479–522.
- Masters, G. & Shearer, P., 1995. Seismic models of the Earth: elastic and anelastic, in *Global Earth Physics: A Handbook of Physical Constants*, pp. 88–103, AGU, Washington.
- Masters, G., Laske, G., Bolton, H. & Dziewonski, A.M., 2000. The relative behavior of shear velocity, bulk sound speed, and compressional velocity in the mantle: implications for chemical and thermal structure, in *Earth's Deep Interior: Mineral Physics and Tomography from the Atomic to the Global Scale*, Vol. 117, pp. 63–87, AGU, Washington.
- Montagner, J.-P., 1994. Can seismology tell us anything about convection in the mantle?, *Rev. Geophys.*, **32**, 115–138.
- Resovsky, J.S. & Trampert, J., 2003. Using probabilistic seismic tomography to test mantle velocity-density relationships, *Earth planet. Sci. Lett.*, 215, 121–134.
- Ritsema, J., van Heijst, H.J. & Woodhouse, J.H., 1999. Complex shear wave velocity structure imaged beneath Africa and Iceland, *Science*, 286, 1925– 1928.
- Ritsema, J., van Heijst, H.J. & Woodhouse, J.H., 2004. Global transition zone tomography, J. geophys. Res., 109, B02302, doi:10.1029/2003JB002610.
- Ritzwoller, M.H. & Lavely, E.M., 1995. Three dimensional seismic models of the Earth's mantle, *Rev. Geophys.*, **33**, 1–66.
- Romanowicz, B., 1991. Seismic tomography of the Earth's mantle, *Ann. Rev. Earth planet. Sci.*, **19**, 77–99.
- Romanowicz, B., 2003. Global mantle tomography: progress status in the past 10 years, *Ann. Rev. Earth planet. Sci.*, **31**, 303–328.
- Shapiro, N.M. & Ritzwoller, M.H., 2002. Monte Carlo inversion for a global shear wave velocity model of the crust and upper mantle, *Geophys. J. Int.*, 151, 88–105.
- Su, W.-J., Woodward, R.L. & Dziewonski, A.M., 1992. Deep origin of midoceanic ridge seismic velocity anomalies, *Nature*, 360, 149–152.
- Su, W.-J., Woodward, R.L. & Dziewonski, A.M., 1994. Degree 12 model of shear velocity heterogeneity in the mantle, *J. geophys. Res.*, 99, 4945– 4980.
- Tanimoto, T., 1988. The three-dimensional shear wave structure in the mantle by overtone waveform inversion, I: radial seismogram inversion, *Geophys. J. R. astr. Soc.*, **89**, 713–740.

Tarantola, A., 1987. Inverse Problem Theory, Elsevier, Amsterdam.

- Tian, Y., Hung, S.-H., Nolet, G., Montelli, R. & Dahlen, F.A., 2007a. Dynamic ray tracing and travel-time corrections for global seismic tomography, J. Comput. Phys., 226(1), 672–687.
- Tian, Y., Montelli, R., Nolet, G. & Dahlen, F.A., 2007b. Computing traveltime and amplitude sensitivity kernels in finite-frequency tomography, *J. Comput. Phys.*, 226(2), 2271–2288.
- Trampert, J., 1998. Global seismic tomography: the inverse problem and beyond, *Inverse Problems*, **14**, 371–385.
- Trampert, J. & Spetzler, J., 2006. Surface wave tomography: finite-frequency effect lost in the null space, *Geophys. J. Int.*, **164**, 394–400.
- Trampert, J. & van der Hilst, R.D., 2005. Towards a quantitative interpretation of global seismic tomography, in *Earth's Deep Mantle: Structure, Composition, and Evolution*, Vol. 160, pp. 47–62, AGU, Washington.
- Trampert, J., Deschamps, F., Resovsky, J.S. & Yuen, D., 2004. Probabilistic tomography maps significant chemical heterogeneities in the lower mantle, *Science*, **306**, 853–856.
- Wang, Z. & Dahlen, F.A., 1995. Validity of surface-wave ray theory on a laterally heterogeneous Earth, *Geophys. J. Int.*, **123**, 757–773.
- Wessel, P. & Smith, W.H., 1995. New version of the generic mapping tools released, EOS, Trans. Am. geophys. Un., 76, 329.
- Woodhouse, J.H. & Dziewonski, A.M., 1984. Mapping the upper mantle: Three-dimensional modelling of Earth structure by inversion of seismic waveforms, *J. geophys. Res.*, **89**, 5953–5986.
- Woodhouse, J.H. & Dziewonski, A.M., 1986. Three-dimensional mantle models based on mantle wave and long period body wave data, *EOS*, *Trans. Am. geophys. Un.*, 67, 307.
- Woodhouse, J.H. & Dziewonski, A.M., 1989. Seismic modelling of the Earth's large-scale three dimensional structure, *Phil. Trans. R. Soc. Lond. A*, 328, 291–308.
- Zhang, Y.-S. & Tanimoto, T., 1992. Ridges, hotspots, and their interpretation as observed in seismic velocity maps, *Nature*, **355**, 45–49.
- Zhao, L. & Dahlen, F.A., 1993. Asymptotic eigenfrequencies of the Earth's normal modes, *Geophys. J. Int.*, 115, 729–758.
- Zhao, L. & Dahlen, F.A., 1995a. Asymptotic normal modes of the Earth, part II: eigenfunctions, *Geophys. J. Int.*, **121**, 585–626.
- Zhao, L. & Dahlen, F.A., 1995b. Asymptotic normal modes of the Earth, part III: Frechet kernel and group velocity, *Geophys. J. Int.*, **122**, 299– 325.
- Zhao, L. & Jordan, T.H., 1998. Sensitivity of frequency-dependent traveltimes to laterally heterogeneous, anisotropic Earth structure, *Geophys. J. Int.*, 133, 683–704.