Models and Fréchet kernels for frequency-(in)dependent $Q$

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SUMMARY
We present a new method for the modelling of frequency-dependent and frequency-independent $Q$ in time-domain seismic wave propagation. Unlike previous approaches, attenuation models are constructed such that $Q$ as a function of position in the Earth appears explicitly as a parameter in the equations of motion. This feature facilitates the derivation of Fréchet kernels for $Q$ using adjoint techniques. Being simple products of the forward strain field and the adjoint memory variables, these kernels can be computed with no additional cost, compared to Fréchet kernels for elastic properties. The same holds for Fréchet kernels for the power-law exponent of frequency-dependent $Q$, that we derive as well. To illustrate our developments, we present examples from regional- and global-scale time-domain wave propagation.

Key words: Tomography; Seismic attenuation; Computational seismology; Theoretical seismology; Wave propagation.

1 INTRODUCTION
Seismic waves propagating through the Earth are attenuated due to a multitude of microscale processes including diffusion and dislocation creep of point defects, grain boundary sliding and the viscous motion in (partially) molten material (e.g. Jackson 2007; Karato 2008, chapter 11). Commonly described macroscopically in terms of the quality factor $Q$, viscoelastic attenuation leads to seismic phase velocity dispersion and to an amplitude reduction of seismic waves (e.g. Dahlen & Tromp 1998; Kennett 2001; Aki & Richards 2002). Numerous laboratory experiments consistently revealed a temperature and frequency dependence of $Q$ for Earth materials that can be described phenomenologically by the Arrhenius-type equation

$$Q(\omega) = \frac{Q_0}{1 + \left(\frac{\omega}{\omega_0}\right)^\alpha} e^{-E/RT},$$

where $E$ is the activation energy, $T$ is temperature, $R$ is the gas constant and $\omega_0$ is a reference frequency (e.g. Goetze 1971; Goetze & Brace 1972; Gueguen et al. 1989; Karato & Spetzler 1990; Jackson 2000). Typical values for the constant $\alpha$, summarized for instance by Karato (2008), range between 0.2 and 0.4. Much of the seismological interest in $Q$ is related to its exponential dependence on $T$, which suggests that attenuation may serve as a proxy for temperature in the Earth. The frequency dependence of $Q$ found in laboratory studies has been confirmed by analyses of seismic data across the seismologically observable frequency band from $\sim10^{-3}$ to $\sim1$ Hz (e.g. Anderson & Minster 1979; Sipkin & Jordan 1979; Flanagan & Wiens 1998; Cheng & Kennett 2002; Lekić et al. 2009). A review on the frequency dependence of $Q$ may be found, for instance, in Romanowicz & Mitchell (2007). Despite convincing evidence for a power-law dependence of $Q$ on frequency, the majority of seismic studies assume frequency-independent attenuation. This simplification can be justified by the sparsity and bandlimited nature of seismic observations that often prevent reliable estimates of $\alpha$.

With the advent of the numerical age, the proper modelling of seismic wave attenuation has received considerable attention. While the implementation of viscoelastic attenuation in frequency-domain numerical modelling is nearly trivial, attenuation is more difficult to implement in time-domain wave propagation schemes that are most frequently used in large-scale 3-D applications (e.g. Igel et al. 1995; Komatitsch & Tromp 1999; Moczo et al. 2002; Chen et al. 2007; Dumbser et al. 2007; Fichtner et al. 2009; Tape et al. 2010). Following the seminal work of Emmerich & Korn (1987) and Carcione et al. (1988a,b), attenuation has been modelled almost exclusively by superpositions of rheological bodies of either Maxwell or Zener type that have been shown to be equivalent (Moczo & Kristek 2005). The discrete ensemble of relaxation mechanisms leads, by construction, to a numerically convenient set of equations (Robertsson et al. 1994; Blanch et al. 1995; van Driel & Nissen-Meyer 2014a). The rheological bodies are described in terms of relaxation variables that are determined such that a prescribed $Q(\omega)$ is matched as closely as possible. The concept of representing a broad absorption band by a superposition of individual relaxation mechanisms already appears in Liu et al. (1976), where it had, however, not been used for numerical modelling.

While the forward problem of viscoelastic wave propagation can be considered solved (at least when sufficient computational resources are available), the inverse problem remains technically challenging because $Q$ does not appear explicitly in the description of
the rheological bodies used to model attenuation (see Section 2.1.1 for details). Instead, \( Q \) is determined implicitly by the set of suitably chosen relaxation parameters. Furthermore, the relation between \( Q \) and the relaxation parameters may vary from one location to another. The absence of an explicit \( Q \) in the time-domain viscoelastic wave equation complicates the computation of Fréchet kernels that are needed to invert for the heterogeneous distribution of \( Q \) in the Earth.

Based on the assumption of a frequency-independent \( Q \), Tramp et al. (2005) proposed to circumvent this problem through the definition of additional adjoint sources for the computation of \( Q \) kernels. This approach, adopted for instance by Bozdag et al. (2011) and Zhu et al. (2013), yields correct kernels, but it also doubles the computational cost because an additional adjoint simulation must be performed. Furthermore, an extension to frequency-dependent \( Q \) seems difficult.

Here we present a new approach to the time-domain modelling of viscoelastic wave propagation with frequency-dependent or frequency-independent attenuation where \( Q \) at a specified frequency appears explicitly in the equations of motion. In addition to improving computational efficiency, this approach allows us to compute Fréchet kernels for \( Q \) and its frequency dependence without the requirement of additional field simulations.

This paper is organized as follows. To introduce basic concepts without heavy notation, we start our developments in Section 2.1 using a 1-D scalar wave equation. In Section 2.1.2, we describe a novel parametrization of attenuation models where \( Q \) appears explicitly. Subsequently, in Section 2.2 we make the transition to the elastic case. A examples, we consider isotropic media, as well as full anisotropy with 21 independent elastic parameters. Section 3 is dedicated to the derivation of shear and bulk \( Q \) kernels, based on the previously derived \( Q \) models. In the interest of a readable text, we defer the derivation of kernels for the power-law exponent \( \alpha \) to the Appendix. Examples of synthetic seismograms for frequency-dependent \( Q \) and corresponding Fréchet kernels are presented in Section 4.

## 2 FORWARD MODELLING

### 2.1 The scalar wave equation

For the purpose of illustration, we start our development with the scalar wave equation. Written in velocity–stress formulation, it consists of the momentum conservation law

\[ \rho \ddot{v} - \nabla \cdot \sigma = f, \tag{2} \]

and the viscoelastic constitutive relation defined by

\[ \dot{\sigma}(t) = \int_{-\infty}^{t} C(t - t') \dot{\varepsilon}(t') \mathrm{d}t'. \tag{3} \]

In eq. (3), \( \sigma \) and \( \varepsilon = \nabla \otimes \varepsilon \) are representative components of the stress tensor \( \sigma \), the elastic tensor \( C \) and the strain tensor \( \varepsilon \), respectively.

#### 2.1.1 Numerical modelling of viscoelastic attenuation

Taking inspiration from Blanch et al. (1995), we model the time-dependence of the elastic modulus \( C \) by a superposition of \( N \geq 1 \) exponential functions with decay times \( \tau_p \) \((p = 1, \ldots, N)\) that phenomenologically mimic different relaxation mechanisms in the Earth:

\[ C(t) = C' \left[ 1 + \tau \sum_{p=1}^{N} D_p^{(p)} e^{-t/\tau_p} \right] H(t). \tag{4} \]

The symbol \( C' \) denotes the relaxed modulus, \( D_p^{(p)} \) are the weights of the relaxation mechanisms, \( H \) is the Heaviside function and \( \tau \) is a parameter that controls the strength of viscoelastic attenuation. As described in Section 2.1.2, the free parameters \( D_p^{(p)}, \tau_p \) and \( \tau \) must be determined such that \( C(t) \) approximates a pre-defined behaviour. Differentiating (4) with respect to time \( t \), and introducing the result into (3), yields

\[ \dot{\sigma} = C'(1 + s \tau) \dot{\varepsilon} + C' \tau \sum_{p=1}^{N} M_p^{(p)}, \quad \text{with} \quad s = \sum_{p=1}^{N} D_p^{(p)}, \tag{5} \]

where the memory variables

\[ M_p^{(p)}(t) = -\frac{D_p^{(p)}}{\tau_p} \int_{-\infty}^{t} e^{-(t-t')/\tau_p} H(t - t') \dot{\varepsilon}(t') \mathrm{d}t'. \tag{6} \]

satisfy the first-order differential equation

\[ \dot{M}_p^{(p)} = -\frac{D_p^{(p)}}{\tau_p} \dot{\varepsilon} - \frac{1}{\tau_p} M_p^{(p)}. \tag{7} \]

The combination of eq. (7), the momentum conservation law (2), and the constitutive relation (3), forms a complete set of equations that describes the propagation of a scalar viscoelastic wave.

#### 2.1.2 Constructing \( Q \) models

The quality factor \( Q(\omega) \) is defined as the ratio

\[ Q(\omega) = \frac{\text{Re} C(\omega)}{\text{IC}(\omega)}, \tag{8} \]

where the complex modulus \( C(\omega) \) is given by

\[ C(\omega) = i \omega \int_{0}^{\infty} C(t) e^{-i\omega t} \mathrm{d}t. \quad (i = \sqrt{-1}). \tag{9} \]

When \( Q \) is sufficiently large, typically \( \geq 100 \), it can be related to the fractional energy loss per oscillation cycle, that is \( \Delta E/E = \frac{2\pi Q}{\omega^2} \). For the specific form of \( C(t) \) defined in eq. (4), we find

\[ Q(\omega) = \left[ 1 + \tau \sum_{p=1}^{N} \frac{D_p^{(p)} \omega^2 \tau_p^{(p)}}{1 + \omega^2 \tau_p^{(p)^2}} \right]^{-1} \sum_{p=1}^{N} D_p^{(p)} \omega^2 \tau_p^{(p)^2} \tag{10} \]

To construct \( Q \)-models that approximate a prescribed empirical frequency dependence of the form suggested already in eq. (1),

\[ Q_{\text{exact}}(\omega) = Q_0 \left( \frac{\omega}{\omega_0} \right)^{\alpha} \tag{11} \]

across the frequencies of interest, we proceed as follows:

(i) We define a set of \( Q_0 \) values, \( Q^{(1)}_0, \ldots, Q^{(M)}_0 \), that span the range of \( Q \) in our earth model.

(ii) We set \( \tau = Q^{(k)}_0 \) for each \( k = 1, \ldots, M \). This defines a collection of numerical \( Q \) models, \( Q(\omega, Q^{(k)}_0) \), via eq. (10).
(iii) We find optimal values for $\tau^{(p)}$ and $D^{(p)}$ by minimizing the cumulative difference between the numerical $Q$ models $Q(\omega)$ and the target $Q$ models $Q_{\text{target}}(\omega)$.

$$J(\tau_p, D_p) = \sum_{k=1}^{M} \left| \left[ \frac{Q^{(k)}(\omega, \omega_0)}{Q^{(k)}(\omega, \omega_0)} - \frac{Q_{\text{target}}^{(k)}(\omega, \omega_0)}{Q_{\text{target}}^{(k)}(\omega, \omega_0)} \right] \right|_\omega. \quad (12)$$

The minimization of $J$ represents a non-linear optimization problem in a low-dimensional parameter space that can be solved efficiently with Monte Carlo-type techniques. Finding optimal $\tau_p$ and $D_p$ for a whole set of $\omega_0$ values has the effect that the approximation

$$Q(\omega) \approx Q_0 \left( \frac{\omega_0}{\omega} \right) ^\alpha \quad (13)$$

effectively holds for any $Q_0$ inside the range of $Q$'s in the earth model. The proposed optimization scheme for the relaxation parameters $\tau^{(p)}$ and $D^{(p)}$ explicitly introduces $Q_0$ into the equations of motion through the enforcement of $\tau = \omega_0^{-1}$ for all relevant $Q_0$ values. There are two immediate advantages of this approach: (i) the search for optimal $\tau^{(p)}$ and $D^{(p)}$ only has to be performed once. Thus, once $\tau^{(p)}$ and $D^{(p)}$ are found, they can be used throughout the earth model even when $Q_0$ is spatially variable. This statement holds provided that $\alpha$ is constant, which includes the frequency-independent case with $\alpha = 0$. (ii) The explicit appearance of $Q_0$ facilitates the computation of Fréchet kernels for $Q_0$ using standard adjoint techniques. The computation of Fréchet kernels for $Q_0$ and $\alpha$ will be described in Section 3 and illustrated in Section 4.

A numerical example for the case of $N = 3$ relaxation mechanisms and a frequency range from 0.02 to 0.2 Hz is shown in Fig. 1. We determined the parameters $\tau^{(p)}$ and $D^{(p)}$ using Simulated Annealing (Kirkpatrick et al. 1983). With $\alpha = 0.3$, $Q(\omega)$ deviates from $Q_{\text{target}}(\omega)$ by less than 3 per cent for values of $\omega_0$ between 50 and 500. While fully sufficient for practical purposes, the accuracy can be improved by using more than three relaxation mechanisms.

### 2.2 Extension to the elastic case

Following the illustrative example for the scalar wave equation in the previous section, we now transition to the fully elastic 3-D case. We consider both general anisotropy (Section 2.2.1) and the practically most relevant isotropic scenario (Section 2.2.2).

#### 2.2.1 General anisotropy

In analogy to eqs (2) and (3), the momentum conservation and viscoelastic stress–strain relation for generally anisotropic media can be written as

$$\rho \ddot{v}_i - \partial_j \sigma_{ij} = f_i \quad (14)$$

and

$$\sigma_{ij}(t) = \sum_{k,l=1, \infty}^3 \int C_{ijkl}(t-t')\dot{\epsilon}_{kl}(t') \, dt'. \quad (15)$$

respectively. The stress–strain relation (15) allows different elastic coefficients $C_{ijkl}$ to be subject to different forms of viscoelastic dissipation. These differences may result in anisotropic attenuation that has been predicted for finely layered media (Carcione 1992; Zhu et al. 2007) and observed in both laboratory and field experiments (e.g. Tao & King 1990; Bao et al. 2012). Anisotropic attenuation in the inner core, with stronger attenuation for waves propagating parallel to the Earth’s spin axis, is also well documented (Craigger 1992; Song & Helmberger 1993). Generalizing eq. (4) for the time dependence of elastic parameters, we have

$$C_{ijkl}(t) = C_{ijkl}^{(0)} \left[ 1 + \tau_{ijkl} \sum_{p=1}^N \int D_{ijkl}^{(p)} e^{-t'/\tau_{ijkl}} \right] H(t). \quad (16)$$

Following the developments in Section 2.1.1, we can eliminate the numerically inconvenient convolutional integral in (15) with the help of memory variables: Introducing the time derivative of (16) into the stress–strain relation (15), yields

$$\dot{\sigma}_{ij} = \sum_{k,l=1}^3 C_{ijkl}(1 + \tau_{ijkl}) \dot{\epsilon}_{kl} + \sum_{k,l=1}^3 C_{ijkl}^{(0)} \tau_{ijkl} \sum_{p=1}^N M_{ijkl}^{(p)}. \quad (17)$$

The memory variables $M_{ijkl}^{(p)}$ defined as

$$M_{ijkl}^{(p)} = -\frac{D_{ijkl}^{(p)}}{\tau_{ijkl}} \int_{-\infty}^\infty e^{-(t-t')/\tau_{ijkl}} H(t-t') \, dt'. \quad (18)$$
satisfy the first-order differential equation
\[ M^{(p)}_{ii} = -\frac{1}{\tau(p)} M^{(p)}_{ii} - \frac{D^{(p)}}{\tau(p)} \dot{e}_{ii}. \]  
(19)

Combined, eqs (14)–(16) constitute a complete set of equations that describes the propagation of dissipative waves in anisotropic media.

2.2.2 The isotropic case

Isotropic media described in terms of the bulk modulus \( \kappa \) and the shear modulus \( \mu \) are the simplest special case of the general viscoelasticity captured in eq. (15). The isotropic elastic tensor is given by
\[
C_{ijkl} = \left( \kappa - \frac{2}{3} \mu \right) \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj}).
\]
(20)

Using the fact that the relaxation parameters \( \tau^{(p)} \) and \( D^{(p)} \) are determined such that the \( \tau_{ijkl} \) from eq. (16) are equal to the inverses of their corresponding \( Q_{ijkl} \), we can write the time-dependent \( C_{ijkl} \) as
\[
C_{ijkl}(t) = \kappa' \delta_{ij} \delta_{kl} \left[ 1 + Q_{0\kappa}^{-1} \sum_{p=1}^{N} D^{(p)} e^{-t/\tau^{(p)}} \right] H(t)
+ \mu' \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj} \right)
\times \left[ 1 + Q_{0\mu}^{-1} \sum_{p=1}^{N} D^{(p)} e^{-t/\tau^{(p)}} \right] H(t).
\]
(21)

Inserting the time derivative of (21) into the stress–strain relation (15) yields the modified stress–strain relation
\[
\dot{\sigma}_{ij} = \kappa' \left[ 1 + Q_{0\kappa}^{-1} x \right] \dot{e}_{ij} + \mu' \left[ 1 + Q_{0\mu}^{-1} x \right] \ddot{e}_{ij}
+ \kappa' Q_{0\kappa}^{-1} \sum_{p=1}^{N} M^{(p)}_{ikl} \delta_{ij} + \mu' Q_{0\mu}^{-1} \sum_{p=1}^{N} \dot{M}^{(p)}_{ikl}.
\]
(22)

where \( \epsilon_{ij} \) and \( \ddot{\epsilon} \) denote the trace and the deviator of the strain tensor \( \epsilon_{ij} \). Similarly, \( M_{ijkl}^{(p)} \) and \( \dot{\dot{M}}_{ijkl}^{(p)} \) are the trace and the deviator of the memory variable tensor \( M_{ijkl}^{(p)} \), defined as in eq. (18). The first two terms in eq. (22) represent a purely elastic stress–strain relation. The last two terms involving the memory variables account for viscoelastic dissipation.

3 SENSITIVITY KERNELS

In Section 2, we established forward problem equations that link viscoelastic parameters to the seismic wavefield. In what follows, we will use this link in order to derive expressions for Fréchet kernels with respect to \( Q \) and \( \alpha \). For this, we assume that a measurement, encoded in the measurement functional \( \chi(\mathbf{u}) \), has been made. Possible measurements include \( L_1 \) and \( L_2 \) waveform differences (Brossier et al. 2009, 2010), cross-correlation traveltimes (Luo & Schuster 1991), generalized seismological data functionals (Gee & Jordan 1992), or various time–frequency misfits (Fichtner et al. 2008). Our analysis rests on the adjoint method, described, for instance, by Tarantola (1984), Tromp et al. (2005), Fichtner et al. (2006a,b) or Chen (2011).

3.1 Adjoint equations and adjoint memory variables

Invoking the adjoint method, the variation \( \delta \chi \) of the measurement functional can be written in terms of the forward wavefield \( \mathbf{u} \), the forward strain tensor \( \epsilon \), the adjoint wavefield \( \mathbf{u}^\dagger \), the adjoint strain tensor \( \epsilon^\dagger \), the variation in density \( \delta \rho \), and the variation of the elastic tensor \( \delta \mathbf{C} \):
\[
\delta \chi = -\int_{-\infty}^{\infty} \int \delta \rho \hat{\epsilon}_{ik}^\dagger(t) \hat{u}_{ik}(t) \, dx \, dt
+ \int_{-\infty}^{\infty} \int \int \hat{\epsilon}_{ik}^\dagger(t) \delta \hat{C}_{ijkl}(t - t') \epsilon_{ij}(t') \, dt' \, dx \, dt.
\]
(23)

The adjoint field is governed by the adjoint equations that may be written in velocity–stress formulation, consisting of the momentum conservation equation
\[
\rho \hat{\epsilon}_{ik}^\dagger = -\delta \sigma_{ik}^\dagger = f_{ik}^\dagger
\]
(24)

and the stress–strain relation
\[
\delta \sigma_{ij}^\dagger = \int \hat{C}_{ijkl}(t - t') \hat{\epsilon}_{ij}^\dagger(t') \, dt'.
\]
(25)

The adjoint source \( f_{ik}^\dagger \) is determined by the definition of the measurement, that is by the specific form of \( \chi \) (see Section 4 for examples). Viscoelastic dissipation in the adjoint stress-strain relation is time-reversed, meaning that current stresses depend on future strains. Since the adjoint equations are, however, solved in reverse time, numerical stability is ensured (Tarantola 1988; Fichtner 2010). Again following the developments in Section 2.1.1, we eliminate the convolutional integral in (25) by defining adjoint memory variables \( M_{ikl}^{(p)} \) as
\[
M_{ikl}^{(p)} = -\frac{D^{(p)}}{\tau^{(p)}} \int_{-\infty}^{\infty} e^{-r(t - t')/\tau^{(p)}} H(t - t') \hat{\epsilon}_{ij}^\dagger \, dt'.
\]
(26)

Differentiating eq. (26) with respect to time \( t \), it follows that the adjoint memory variables satisfy the first-order differential equation
\[
M_{ikl}^{(p)} = \frac{1}{\tau^{(p)}} M_{ikl}^{(p)} + \frac{D^{(p)}}{\tau^{(p)}} \dot{M}_{ikl}^{(p)}.
\]
(27)

The resulting modified stress–strain relation for the adjoint field is
\[
\delta \sigma_{ij}^\dagger = C_{ijkl}^\dagger (1 + \tau_{ijkl}) \hat{\epsilon}_{ij}^\dagger + \sum_{k,l=1}^{3} C_{ijkl}^\dagger \tau_{ijkl} \sum_{p=1}^{N} M_{ikl}^{(p)}.
\]
(28)

Equipped with the complete set of adjoint equations, consisting of eqs (23), (27) and (28), we can proceed with the calculation of sensitivity kernels for \( Q \) and \( \alpha \). In the interest of a lighter notation, we will consider shear and bulk \( Q \) separately, and we transfer the detailed derivation of \( \alpha \) kernels to the Appendix.

3.2 Shear \( Q \)

Restricting ourselves to an isotropic medium with \( C_{ijkl} = (\kappa - \frac{2}{3} \mu) \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \delta_{il} \delta_{kj} \) and variations in the shear modulus \( \mu \), eq. (23) condenses to
\[
\delta \chi = 2 \int_{-\infty}^{\infty} \int \left[ \int \hat{\epsilon}_{ij}^\dagger(t) \delta \mu(t - t') \, dt' \right] \hat{\epsilon}_{ij}(t) \, dx \, dt,
\]
(29)
where \( \hat{\varepsilon}_{ij} \) and \( \hat{\chi}_{ij} \) are the deviatoric parts of \( \varepsilon_{ij} \) and \( \chi_{ij} \), respectively. Invoking the chain rule, we can express \( \delta\mu \) in (29) in terms of variations in \( Q_{0\mu} \):

\[
\delta \mu = \frac{\partial \mu}{\partial Q_{0\mu}} \delta Q_{0\mu} .
\]  

(30)

The partial derivatives \( \partial \mu / \partial Q_{0\mu} \) follows from the definition of the time-dependent elastic modulus in (4), with the general relaxed modulus \( C' \) set equal to the relaxed shear modulus \( \mu' \):

\[
\frac{\partial \mu(t)}{\partial Q_{0\mu}} = -\mu' Q_{0\mu}^{-2} \left[ \sum_{p=1}^{N} D^{(p)} e^{-t/\tau^{(p)}} \right] H(t) .
\]  

(31)

Using (30) and (31), we can reformulate the integral over \( \hat{\varepsilon}_{ij}'(t) \delta \mu(t - t') \) that appears in eq. (29):

\[
\int_{-\infty}^{\infty} \hat{\varepsilon}_{ij}'(t) \delta \mu(t - t') \, dt = \int_{-\infty}^{\infty} \hat{\varepsilon}_{ij}'(t) \delta \mu(t - t') \, dt
\]  

\[
= -\mu' Q_{0\mu}^{-2} \sum_{p=1}^{N} \int_{-\infty}^{\infty} D^{(p)} e^{-t/\tau^{(p)}} H(t - t') \hat{\varepsilon}_{ij}'(t) \delta Q_{0\mu} \, dt .
\]  

(32)

Identifying copies of the adjoint memory variables \( M^{(p)}_{ikj} \), defined in (26), we can condense (32) into

\[
\int_{-\infty}^{\infty} \hat{\varepsilon}_{ij}'(t) \delta \mu(t - t') \, dt = \mu' Q_{0\mu}^{-2} \sum_{p=1}^{N} \tau^{(p)} \tilde{M}^{(p)}_{ij} \delta Q_{0\mu} .
\]  

(33)

We can now combine (29) with (33) in order to write \( \delta \chi \) in terms of the volumetric Fréchet or sensitivity kernel \( K_{Q\mu} \):

\[
\delta \chi = \int_{V} K_{Q\mu}(x) \delta \ln Q_{0\mu}(x) \, dx ,
\]  

(34)

where \( K_{Q\mu} \) can be explicitly computed from the interaction of the forward strain deviator \( \hat{\varepsilon}_{ij} \) and the deviator of the adjoint memory variables, \( \tilde{M}_{ikj} \):

\[
K_{Q\mu} = 2 \mu' Q_{0\mu}^{-1} \sum_{p=1}^{N} \tau^{(p)} \int_{-\infty}^{\infty} \tilde{M}^{(p)}_{ij} \hat{\varepsilon}_{ij} \, dt .
\]  

(35)

Eq. (35) reveals that the kernel for \( Q_{0\mu} \) can be computed in a similar fashion as kernels for elastic parameters, velocity and density; without any additional computational requirements. The adjoint equations are solved in reversed time which automatically yields the adjoint memory variables needed to evaluate the time integrals in (35). For comparison, the sensitivity kernel for the shear modulus \( \mu \) in a non-dissipative medium is given by (e.g. Tromp et al. 2005; Fichtner 2010)

\[
K_{\mu} = 2 \mu \int_{-\infty}^{\infty} \hat{\varepsilon}_{ij} \, dt .
\]  

(36)

Thus, for the computation of the \( Q \) kernel \( K_{Q\mu} \), the term \( Q_{0\mu}^{-1} \sum_{p=1}^{N} \tau^{(p)} \tilde{M}^{(p)}_{ij} \), involving the deviator of the adjoint memory variables, simply takes the place of the adjoint strain tensor \( \hat{\varepsilon}_{ij} \) in eq. (36). Following similar steps as above, we show in the Appendix that the Fréchet kernel \( K_{\alpha} \) for the exponent \( \alpha \) in the power-law frequency dependence of \( Q_{0\mu} \) (eq. 11), is given by

\[
K_{\alpha} = -2 \mu' \alpha Q_{0\mu}^{-1} \sum_{p=1}^{N} \tau^{(p)} D^{(p)} \int_{-\infty}^{\infty} \tilde{M}^{(p)}_{ij} \hat{\varepsilon}_{ij} \, dt .
\]  

(37)

### 3.3 Bulk \( Q \)

Considering only variations in the viscoelastic properties related to the bulk modulus \( \kappa \), the variation of the elastic tensor \( C_{ikj} \) reduces to \( \delta C_{ikj} = \delta \kappa \). The variation of the misfit or measurement functional \( \chi \) can then be written in terms of the traces \( \epsilon_{kk} \) and \( \chi_{kk} \) of the forward and adjoint strain tensors:

\[
\delta \chi = \int_{V} K_{Q\kappa}(x) \delta \ln Q_{0\kappa}(x) \, dx ,
\]  

(39)

with the sensitivity kernel

\[
K_{Q\kappa} = \kappa' Q_{0\kappa}^{-1} \sum_{p=1}^{N} \tau^{(p)} \int_{-\infty}^{\infty} M^{(p)}_{kk} \epsilon_{kk} \, dt .
\]  

(40)

In eq. (40), \( M^{(p)}_{kk} \) denotes the trace of the memory variable tensor \( M^{(p)}_{ij} \). Just as the kernel for shear \( Q \) in eq. (35), the kernel for bulk \( Q \) can be computed from the forward strain and the adjoint memory variables that are a natural by-product of the adjoint solution. Again, for comparison, we note the kernel for the bulk modulus \( \kappa \) in a non-dissipative medium is given by (e.g. Tromp et al. 2005; Fichtner 2010)

\[
K_{\kappa} = \kappa \int_{-\infty}^{\infty} \epsilon_{kk} \, dt .
\]  

(41)

It follows that a simple replacement of \( \epsilon_{kk} \) by \( Q_{0\kappa}^{-1} \sum_{p=1}^{N} \tau^{(p)} M^{(p)}_{kk} \) in eq. (40) yields \( Q \) instead of \( \kappa \) kernels.

In the Appendix, we demonstrate that the Fréchet kernel for the power-law exponent of bulk \( Q \) is given by

\[
K_{\alpha} = -2 \mu' \alpha Q_{0\kappa}^{-1} \sum_{p=1}^{N} \tau^{(p)} \int_{-\infty}^{\infty} M^{(p)}_{kk} \epsilon_{kk} \, dt .
\]  

(42)

### 4 EXAMPLES

To illustrate the practical implementation of frequency-dependent \( Q \) and its effect on seismic waveforms, as well as the computation of \( Q \) and \( \alpha \) kernels, we present various examples from global- and regional-scale wave propagation.

#### 4.1 Global wave propagation

In order to model global seismic wave propagation in a broad frequency range, while keeping the computational requirements at a manageable level, we limit ourselves to the radially symmetric
dependence of ranging from 2 to 200 s. The example illustrates that the frequency-values of a small collection of which is presented in Fig. 3 for a period band ence frequency lead to notable differences in synthetic seismograms, hand panel of Fig. 2. The original, frequency-independent & Anderson 1981), modified such that following sections. As earth model we again use PREM (Dziewo ´nski & Anderson 1981). Being a special case of an axisymmetric medium, the equations of motion for PREM can be reduced to a system of PDE’s in two space variables, and solved efficiently by the time-domain spectral-element code AxiSEM (Nissen-Meyer et al. 2007, 2014; van Driel & Nissen-Meyer 2014b).

We consider three different Q models, summarized in the left-hand panel of Fig. 2. The original, frequency-independent Q of PREM (black curve), a frequency-dependent Q with α = 0.3 and reference frequency 1 Hz (red curve), and a frequency-dependent Q with α = 0.3 and reference frequency 0.1 Hz (blue curve). The phase velocities are matched to the phase velocities of PREM at the central period of the numerical simulation, which is 22 s. The resulting phase velocity dispersion curves are shown in the right-hand panel of Fig. 2.

The frequency dependence of Q and the different choice in reference frequency lead to notable differences in synthetic seismograms, a small collection of which is presented in Fig. 3 for a period band ranging from 2 to 200 s. The example illustrates that the frequency-dependence of Q is generally not a small effect because realistic values of α between 0.2 and 0.4 (e.g. Karato 2008) can lead to substantial modification of Q away from the reference frequency.

4.2 Regional-scale wave propagation

In our next example, we consider wave propagation at regional scales, that is over distances of few hundred kilometres. Our computational domain, shown in the upper right-hand panel of Fig. 4, is centred on Turkey. We locate the earthquake in eastern Turkey and choose an explosion as source mechanism in order to exclude radiation pattern effects on the sensitivity kernels computed in the following sections. As earth model we again use PREM (Dziewo ´nski & Anderson 1981), modified such that Qv and Qs are frequency dependent as shown in Fig. 1. For the simulation of 3-D seismic wave propagation, we use the spectral-element solver SES3D, described in Fichtner & Igel (2008) and Fichtner et al. (2009).

A comparison of synthetic seismograms with and without attenuation is shown in Fig. 4 for station ADVT, located in western Turkey at an epicentral distance of 9°. As a result of the short epicentral distance and the short dominant period of 8 s, the wavefield mostly senses crustal and uppermost mantle structure where Qv and Qs in PREM range around 600 and 58 000, respectively. It follows that the effects on phase and amplitude are small, but noticeable. The amplitudes of both body and surface waves are reduced by around 5 per cent. The time shifts induced by the presence of viscoelastic attenuation range around 1 s.

For the calculation of Fréchet kernels we limit ourselves to two types of measurements: (1) Relative L2 amplitude differences are defined as

\[ \chi = A = \frac{\int \frac{u - u_0}{u_0} \, dt}{\int \frac{u_0^2}{u_0} \, dt}, \]

with u and u0 denoting synthetic and observed seismograms, respectively. For our examples, we restrict ourselves to the vertical components, that is u = uz. (2) Correlation traveltime shifts are defined as the time T where the correlation between observed and synthetic waveforms reaches its maximum (e.g. Luo & Schuster 1991; Dahlen et al. 2000):

\[ \chi = T = \arg \max \int u(t) u_0(t + \tau) \, d\tau. \]

The adjoint sources f† for these measurements, that is the right-hand sides of the adjoint eq. (24), are given by

\[ f_{\alpha}^*(t) = \frac{2u(t)}{\int u^2 \, dt} e_\alpha, \]

and

\[ f_{\mu}^*(t) = -\frac{\tilde{u}(t)}{\int \tilde{u}^2 \, dt} e_\mu, \]

respectively (e.g. Luo & Schuster 1991; Fichtner 2010). Equipped with eqs (45) and (46), we can solve the adjoint equations that provide the adjoint memory variables needed to compute Fréchet kernels for viscoelastic parameters, according to eqs (35) and (40).

4.2.1 Fréchet kernels for Q at the reference frequency

Fig. 5 displays horizontal slices through Fréchet kernels for Qμv, that is the shear Q at the reference circular frequency ω0. Kernels are computed based on eq. (35) and for measurements performed in three different time windows. While the time window from 246 to 333 s mostly contains higher-mode Rayleigh waves, the time window from 362 to 400 s is dominated by the fundamental-mode Rayleigh wave. Acknowledging that more elaborate measurements—based for instance on multitapers or various time-frequency transforms (e.g. Laske & Masters 1996; Zhou et al. 2004; Fichtner et al. 2008)—are possible, we do not apply additional
Figure 3. Comparison of vertical-component displacement seismograms (bandpass filtered between 2 and 200 s period) for a moment magnitude $M_w = 5.0$ event in 126 km depth under the Tonga islands, computed with AxiSEM in the anisotropic PREM model without ocean with the three different attenuation models shown in Fig. 2. The traces are plotted for the GSN stations indicated in the map. The zoom windows are indicated with red rectangles in the record section and the timescale is relative to the ray-theoretical arrival.
Figure 4. Comparison of vertical-component synthetic seismograms without (black) and with (red) viscoelastic dissipation for a dominant period of 8 s. The source–receiver configuration is shown to the right-hand side, with the computational domain shaded in light grey. As earth model, we use the spherically symmetric PREM (Dziewonski & Anderson 1981) with a frequency-dependent $Q$ constructed as in Fig. 1. The zoom into the $P$ wave and surface wave trains (lower left- and lower right-hand side, respectively), reveals time shifts of around 1 s and amplitude variations on the order of 5 per cent.

Figure 5. Horizontal slices through Fréchet kernels for relative perturbations in $Q_{0\mu}$ for measurements in different time windows on the vertical-component velocity seismogram from station ADVT (see Fig. 4). The time windows are indicated in the top row by grey shading. Kernels for amplitude and traveltime measurements are shown in the second and third row, respectively. All kernels are plotted at the depth where they attain their largest values. Note the different colour scales.

Filters or time windows in order to keep the examples illustrative and repeatable. Kernels for amplitude and traveltime measurements are shown in the second and third rows of Fig. 5, respectively. All kernels are plotted at the depth where they attain their maximum values.

Being a composite of various higher modes, the time window from 246 to 333 s yields Fréchet kernels that deviate from the simple cigar shape produced by the fundamental-mode Rayleigh wave in the 362–400 time window. The comparatively high frequencies in the time window from 333 to 362 s lead to a thinner Fresnel
zone than for the other time windows where the dominant frequencies are lower. Generally, amplitude and traveltime measurements have similarly strong sensitivity to relative perturbations in $Q_0\kappa$, as previously noted, for instance, by Zhou (2009).

Fréchet kernels for $Q_0\kappa$, that is bulk $Q$ at the reference frequency, can be computed using eq. (40). Kernels for the same measurement windows and measurements as in Fig. 5 are shown in Fig. 6. As expected for surface waves with little sensitivity to the bulk modulus, sensitivities for bulk $Q$ are several orders of magnitude smaller than for shear $Q$. The overall geometrical pattern, however, remains unchanged.

4.2.2 Fréchet kernels for the power-law exponent $\alpha$

The computation of Fréchet kernels for $\alpha$, that is the power-law exponent in the frequency dependence of $Q$, requires knowledge
of the partial derivatives $\partial D(\alpha) / \partial \alpha$ (see eqs 37 and 42). Since the weights $D(\alpha)$ are computed by numerical optimization as outlined in Section 2.1.2, their partial derivatives are not explicitly available. They can, however, be approximated by computing weights $D(\alpha + \delta \alpha)$ for a slightly perturbed power-law exponent $\alpha$:

$$\frac{\partial D(\alpha)}{\partial \alpha} \approx \frac{D(\alpha + \delta \alpha) - D(\alpha)}{\delta \alpha}. \quad (47)$$

For our example with three relaxation mechanisms, the finite-difference approximation (47) yields the values $\partial D(\alpha) / \partial \alpha = -3.06$, $\partial D(\alpha) / \partial \alpha = -1.54$ and $\partial D(\alpha) / \partial \alpha = 2.56$ for shear $Q$.

Fréchet kernels for fractional perturbations in shear $Q$, displayed in Fig. 7 for the previously used time windows and measurements, are orders of magnitude smaller than kernels for fractional perturbations in shear $Q$. While more targeted measurements are possible (e.g. Cheng & Kennett 2002; Lekić et al. 2009; Kennett & Abdullah 2011), this result still reflects that the frequency-dependence of $Q$ in the Earth is difficult to constrain.

5 DISCUSSION AND CONCLUSIONS

We presented a novel method for the modelling of frequency-dependent and frequency-independent $Q$ in time-domain numerical wave propagation. In contrast to previous approaches (e.g. Emmerich & Korn 1987; Carcione et al. 1988a,b; Blanch et al. 1995), $Q$ as a function of position in the Earth is introduced explicitly into the equations of motion.

A key element of our method is the determination of only one set of relaxation parameters $\tau(\alpha)$ and $D(\alpha)$ from eq. (4) that is valid for the full range of $Q_0$ values in the earth model. This is different from more classical approaches where a set of relaxation parameters is determined individually for each $Q_0$ value (e.g. Emmerich & Korn 1987; Blanch et al. 1995; van Driel & Nissen-Meyer 2014a). A direct consequence of working with one universal set of relaxation parameters are larger discrepancies between the target $Q$ model and the actual numerical $Q$ model. For most practical purposes, however, these errors are hardly relevant. Using, for instance, $N = 3$ relaxation mechanisms for frequencies between 0.02 and 0.2 Hz, and $Q_0$ between 50 and 500, the relative errors between the target $Q$ and the numerical $Q$ shown in Fig. 1 are below 3 per cent. This error is well below lateral variations of shear $Q$ in global models that are on the order of ±100 per cent (e.g. Romanowicz 1995; Selby & Woodhouse 2002; Warren & Shearer 2002; Gung & Romanowicz 2004; Dalton et al. 2008). Differences between 1-D $Q$ models typically range between 10 and 100 per cent (e.g. Dziewoinski & Anderson 1981; Widmer et al. 1991; Durek & Ekström 1996; Resovsky et al. 2005; Trampert & Fichtner 2013).

The most relevant tuning parameters in our approach are the number and values of the target $Q_0(\alpha)$, as well as the number of relaxation parameters. While one should ideally give a generally valid recipe for the perfect distribution of the target $Q_0(\alpha)$, we think that carefully conducted numerical experiments with different choices for $Q_0(\alpha)$ are more likely to provide good results for specific applications with their specific requirements. The same holds for the number of relaxation mechanisms. The approximation can be improved through the incorporation of additional relaxation mechanisms, though at the expense of increase computational costs.

The $\alpha$ kernels derived in the Appendix and computed in Section 4.2.2 for example measurements, in principle provide a tool that enables inversions for the frequency-dependence of $Q$ as a function of position. Since $Q$ itself tends to be poorly resolved (e.g. Resovsky et al. 2005), a good spatial resolution of $\alpha$, that is comparable to the spatial resolution of seismic velocities, seems unlikely. Model basis functions for $\alpha$ will thus need to have a broader spatial extent, or even be constant for the whole Earth—depending on the resolving power of a specific data set. In our description of $Q$, we so far assumed a constant $\alpha$ throughout the Earth. In the case of spatially variable $\alpha$, this aspect would need to be relaxed, and position dependent weight factors $D(\alpha)$ would need to be determined.

The most important advantage of our approach lies in the computationally efficient calculation of Fréchet kernels that does not require additional computational costs, compared to the calculation of Fréchet kernels for elastic properties. Fréchet kernels for anelastic properties can generally be expressed in terms of the forward strain field and the adjoint memory variables that are a by-product of any adjoint calculation in a viscoelastic medium.

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REFERENCES


APPENDIX A: COMPUTING $\alpha$-KERNELS

In this Appendix, we provide a detailed derivation of the Fréchet kernels for the exponent $\alpha$ in the power-law frequency dependence of $Q$, as defined in eq. (11). For this we first note that for $Q_0 \gg 1$, eq. (13) can be transformed to

$$ \sum_{p=1}^{N} \frac{D_p \alpha_p}{1 + \alpha_p^2 \tau_p^2} \approx \left( \frac{\alpha_0}{\alpha} \right)^{\alpha}. $$

(A1)

Keeping the relaxation times for a specific target frequency range fixed, eq. (A1) implies that the vector of weights $D = (D_1^1, \ldots, D_N^N)^T$ only depends on $\alpha$ and not on $Q_0$, that is $D = D(\alpha)$. Equipped with this result, we now proceed with the calculation of $\alpha$ kernels. In the interest of a lighter notation, we again consider shear and bulk attenuation separately.

A1 Shear attenuation

In isotropic media with $C_{ijkl} = (\kappa - \frac{2}{3} \mu) \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \delta_{il} \delta_{jk}$ and variations in the shear modulus $\mu$, the variation of the measurement functional (eq. 24) takes the form

$$ \delta \chi = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \tilde{e}_{ij}^\tau(t) \tilde{\mu}(t - \tau) d\tau \right] \tilde{\epsilon}_{ij}^\tau(t') d\tau' d\tau', $$

(A2)

where $\tilde{\epsilon}_{ij}$ and $\tilde{\epsilon}_{ij}^\tau$ are the deviatoric parts of $\epsilon_{ij}$ and $\epsilon_{ij}^\tau$, respectively. In the next step, we express $\delta \mu$ in (29) in terms of variations in $\alpha$. For this, we invoke the chain rule and the previously noted fact that the weights $D_p$ only depend on $\alpha$ (eq. A1):

$$ \delta \mu = \sum_{p=1}^{N} \frac{\partial \mu}{\partial D_p^p} \frac{\partial D_p^p}{\partial \alpha} \delta \alpha. $$

(A3)

The partial derivatives $\partial \mu / \partial D_p^p$ follow from the definition of the time-dependent elastic modulus in (4), with the general relaxed modulus $C$ set equal to the relaxed shear modulus $\mu^*:

$$ \frac{\partial \mu(t)}{\partial D_p^p} = \mu^* Q_0^{-1} \frac{\partial D_p^p}{\partial \alpha} e^{-\tau^p(t)} H(t). $$

(A4)

Using (A3) and (A4), we can reformulate the integral over $\tilde{e}_{ij}^\tau(t) \tilde{\mu}(t - \tau)$ that appears in eq. (A2):

$$ \int_{-\infty}^{\infty} \tilde{e}_{ij}^\tau(t) \tilde{\mu}(t - \tau) d\tau = \int_{-\infty}^{\infty} \tilde{e}_{ij}^\tau(t) \delta \mu(t - \tau) d\tau $$

$$ = \mu^* Q_0^{-1} \sum_{p=1}^{N} \frac{\partial D_p^p}{\partial \alpha} e^{-\tau^p(t)} H(t - \tau) \tilde{e}_{ij}^\tau(t) \delta \alpha d\tau. $$

(A5)

Substituting the adjoint memory variables $M_{ij}^{p(t)}$, defined in (26), we can simplify (A5) into

$$ \int_{-\infty}^{\infty} \tilde{e}_{ij}^\tau(t) \delta \mu(t - \tau) d\tau = -\mu^* Q_0^{-1} \sum_{p=1}^{N} \frac{\partial D_p^p}{\partial \alpha} D_p^p \int_{-\infty}^{\infty} M_{ij}^{p(t)} \tilde{e}_{ij}^\tau(t) \delta \alpha d\tau. $$

(A6)

Combining eqs (A2) with (A6) we can write the variation of the measurement functional $\delta \chi$ in terms of a volumetric Fréchet or sensitivity kernel:

$$ \delta \chi = \int \mathcal{K}_\alpha(x) \delta \ln \alpha(x) dx, $$

(A7)

where the kernel $\mathcal{K}_\alpha$ is given in terms of the forward strain deviator $\tilde{\epsilon}_{ij}$ and the deviator of the adjoint memory variables, $M_{ij}^{p(t)}$:

$$ \mathcal{K}_\alpha = -2\mu^* Q_0^{-1} \sum_{p=1}^{N} D_p^p \frac{\partial D_p^p}{\partial \alpha} \int_{-\infty}^{\infty} M_{ij}^{p(t)} \tilde{e}_{ij}^\tau(t) d\tau. $$

(A8)

This proofs eq. (37).

A2 Bulk attenuation

For variations in the viscoelastic properties related to the bulk modulus $\kappa$, the variation of the elastic tensor $C_{ijkl}$ condenses to $\delta C_{ijkl} = \delta \kappa$. The variation of the measurement functional $\chi$ can then be written in terms of the traces $\epsilon_{kk}$ and $\epsilon_{kk}^\tau$ of the forward and adjoint strain tensors:

$$ \delta \chi = \int \int \int \int \epsilon_{kk}^\tau(t) \delta \mu(t - \tau) d\tau \int \epsilon_{kk}^\tau(t') d\tau'. $$

(A9)

Following exactly the same steps as in Section A1, we transform (A9) into

$$ \delta \chi = \int \mathcal{K}_\kappa(x) \delta \ln \alpha(x) dx, $$

(A10)

with the Fréchet kernel

$$ \mathcal{K}_\kappa = -\kappa^* Q_0^{-1} \sum_{p=1}^{N} D_p^p \frac{\partial D_p^p}{\partial \alpha} \int_{-\infty}^{\infty} M_{kk}^{p(t)} \epsilon_{kk}^\tau(t) d\tau. $$

(A11)

This is the result previously stated without proof in eq. (42).

References


