Tutorials mantle convection

Parameterized models for planetary thermal evolution

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1 planetology/computerlabs/Labs2017/Doc/thermal_evol_mod
1 Introduction

Planetary thermal evolution is controled through thermal convection by solid state creep flow in the high viscosity silicate mantle of the terrestrial planets. Most direct geophysical evidence for large scale convective flow in the Earths mantle comes from the observation of the global distribution of heatflow which is focussed near mid-ocean ridges and strongly correlated with the solidification age of the oceanic basaltic crust which increases with the distance from the ridge.

These observations are now explained by large scale flow in the mantle associated with thermal convection driven both by internal heating and secular cooling. In the following section several models are developed for the secular cooling history of a planet.

In the following several relatively simple models for the thermal evolution of a planet are presented. These models describe the evolution of relevant global quantities like the volume averaged tempearture and the heat flux through the surface.

2 Conservation principle of thermal energy

An approximation of the cooling behavior of a (planetary) body contained in a volume Ω, can be obtained from a conservation principle for thermal energy by integrating over the model domain with volume $V = \int_{\Omega} dV$ with closed surface $S$ with surface area $A = \int_{S} dA$, \[ C \frac{dT}{dt} = -Q(t) + \langle \rho H(t) \rangle \] (2) is an ordinary differential equation (ODE) for the time dependent volume averaged temperature of the cooling body.

problem 1 Determine the physical dimensions (SI units) of the different terms in (2).

In the literature on planetary thermal evolution the ratio of the instantaneous internal heatproduction rate and the surface heat flux is denoted by the non-dimensional Urey number,

$$ Ur(t) = \frac{\int_{\Omega} \rho H(t) dV}{\int_{S} q(t) dA} = \frac{\langle \rho H(t) \rangle}{\langle q \rangle} = \frac{\langle \rho H(t) \rangle}{Q(t)} $$

problem 2 Verify that a unit Urey number implies a stationary volume average temperature.

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3 See also lecture notes of a course in Geodynamics, Physics of the Earth’s interior, (www.geo.uu.nl/~berg/geodynamics).
In general the heatflow density \( q(x,t) \) varies with the internal temperature \( \langle T \rangle \) and hence \( Q(t) \) is not known a priori and assumptions concerning the relation between surface heat flux and internal temperature have to be made in order to make (2) solvable. In the following we consider several parameterizations for \( Q \) in terms of \( \langle T \rangle \).

### 2.1 A constant heat flux approximation

A first rough estimate of the thermal evolution of the cooling body can be obtained by assuming the surface heat flux to be constant in time and neglecting internal heating.

**problem 3** Determine the thermal history \( \langle T(t) \rangle \) for this constant heatflow model by solving the differential equation (2), assuming \( Q(t) = Q_0 \) (constant) and an initial temperature \( \langle T(0) \rangle = T_0 \). The Kelvin time \( \tau_K \) of this system is defined as the time required to remove all heat from the system. Compute the Kelvin time for the following approximate earthlike parameters: planetary radius \( R = 6371 \) km, mean density \( \rho_0 = 5500 \) kg/m\(^3\), mean specific heat \( c_P = 1000 \) W/K/kg, surface heat flux \( 44 \times 10^{12} \) W, and an initial temperature of 3000K.

### 2.2 A linear heat flux approximation

In a refined approximation of the surface heat flux parameterization we assume a linear relationship between \( Q \) and \( \langle T \rangle \),

\[
Q(t) = \alpha_Q \langle T(t) \rangle
\]  

resulting in the analytical solution,

\[
\langle T(t) \rangle = \langle T(0) \rangle \exp(-t/\tau_T) + \exp(-t/\tau_T) \int_0^t \exp(t'/\tau_T) \langle \rho H(t') \rangle \rho c_p dt'
\]  

where \( \langle T(0) \rangle \) is the initial temperature value and the thermal relaxation time \( \tau_T \) is defined as,

\[
\tau_T = \frac{C}{\alpha_Q} = \frac{C \langle T(t) \rangle}{Q(t)}
\]

**problem 4** Verify the solution (5) by inspection, using the expression for the derivative of an integral,

\[
\frac{d}{dt} \int_0^t f(\tau) d\tau = f(t)
\]

For reasonable present day Earth values of the parameters in Table 1, (5,6) can be used to compute a thermal history.
Table 1: Estimates of present day values of physical parameters for Earth.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>Earth’s outer radius</td>
<td>$6.371 \times 10^6$</td>
<td>$m$</td>
</tr>
<tr>
<td>$R_C$</td>
<td>Core radius</td>
<td>$3.486 \times 10^6$</td>
<td>$m$</td>
</tr>
<tr>
<td>$h = R - R_C$</td>
<td>mantle depth</td>
<td>$2.885 \times 10^6$</td>
<td>$m$</td>
</tr>
<tr>
<td>$\Delta V_M = \frac{3R^2}{(R^3 - R_C^3)}$</td>
<td>mantle surface/volume ratio</td>
<td>$5.631 \times 10^{-7}$</td>
<td>$m^{-1}$</td>
</tr>
<tr>
<td>$\langle T \rangle$</td>
<td>Average Temperature</td>
<td>$2.2 \times 10^3$</td>
<td>$K$</td>
</tr>
<tr>
<td>$\langle q \rangle$</td>
<td>Heatflow density</td>
<td>$87.2 \times 10^{-3}$</td>
<td>$W m^{-2}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Mantle Density</td>
<td>$5.19 \times 10^3$</td>
<td>$kg m^{-3}$</td>
</tr>
<tr>
<td>$c_P$</td>
<td>Specific heat</td>
<td>$1.25 \times 10^3$</td>
<td>$JK^{-1}kg^{-1}$</td>
</tr>
<tr>
<td>$H(t) = H(0) \exp(-t/\tau_H)$</td>
<td>internal heating rate</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$H(0)$</td>
<td>initial value int. heating rate</td>
<td>-</td>
<td>$W kg^{-1}$</td>
</tr>
<tr>
<td>$H(t \approx 4.5Gyr)$</td>
<td>current internal heating</td>
<td>$5.5 \times 10^{-12}$</td>
<td>$W kg^{-1}$</td>
</tr>
<tr>
<td>$\tau_H$</td>
<td>radioactive decay time</td>
<td>$\approx 4 \times 10^9$</td>
<td>$yr$</td>
</tr>
<tr>
<td>$\tau_T$</td>
<td>thermal relaxation time</td>
<td>$2.53 \times 10^7$</td>
<td>$s$</td>
</tr>
<tr>
<td>$\alpha_Q = C/\tau_T$</td>
<td>heat flux coefficient</td>
<td>$8.02 \times 10^9$</td>
<td>$yr$</td>
</tr>
</tbody>
</table>

problem 5  
Equation (2) is in dimensional SI units. It is more convenient to use scale values for temperature say $T_0 = 1000 \ K$ and time $t_0 = 1 \ Gyr$. This way the new variables have values of order of magnitude 1. The new non-dimensionalized variables, denoted by primes, $t'$ and $T'$ are then defined in terms of their scale values as, $t = t' t_0$ and $\langle T \rangle = T' T_0$.

Apply this scaling to the ODE (2), using the linearized heat flux (4) to obtain the non-dimensional ODE,

$$\frac{dT'}{dt'} = -a T' + bH$$

(8)

where the coefficients are defined as $a = \frac{6}{C} \alpha_Q$ and $b = \frac{\rho C}{\rho t_0}$. This standard form of the ODE can be solved using numerical integration methods.

3 Parameterized convection models

For a planetary mantle, subject to thermal convection by solid state creep flow, the combined heat flux through the outer surface and the core mantle boundary are part of the conservation equation for thermal energy (1). Thermal convection in a planetary mantle is modelled as Rayleigh-Benard (R-B) convection of a highly viscous fluid layer heated from below and cooled from the top. In the following we present parameterizations of the heat flux in terms of the vigor of convection in the planetary mantle. First we consider a model where only the heat flux through the outer surface is taken into account.

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An introduction to numerical integration of ODE’s can be found in the course notes: Programming and modelling, Module 3, Fortran programming (part 3), Applied to numerical solution of differential equations for planetary orbits. (www.geo.uu.nl/~berg/progmod3)
3.1 Parameterization of the heat flow through the outer surface

To this end the total surface heat flux term \( Q \) in (2) is expressed in the average heatflow density through the outer surface \( Q = Qs = \langle q \rangle A/V = q_s A/V \), \( A \) the area of the outer surface. This is then expressed in the non-dimensional Nusselt number, i.e. a measure of convective heat transport. The Nusselt number \( Nu \) is defined as the ratio of the actual mean surface heatflow density and a scale value usually defined as the value for a purely conductive (non-convecting) reference case.

\[
Nu = \frac{q_s}{k\Delta T/h}
\]  

(9)

where \( k \) is the coefficient of thermal conductivity (Table 2) and \( \Delta T \) is the positive temperature contrast between the hot bottom \( T_b = T_s + \Delta T \) and cold top boundary \( T = T_s \) of a corresponding Rayleigh-Benard (R-B) convection cell (see below).

The model equation describing the evolving internal temperature of the cooling body can now be rewritten in terms of the Nusselt number and the resulting formulation can be used for implementations in computer software for the numerical solution of the model equation. By specifying different models through their specific definition of the Nusselt number a general implementation of the parameterized convection model is obtained.

Here we consider a simple case corresponding to the Boussinesq-Approximation (BA) of the full R-B convection equations. This type of convection is characterized by convection cells with hot bottom- and cold (top) thermal boundary layers and a well mixed approximately isothermal interior layer. The horizontally averaged temperature profile is anti-symmetric with respect to the temperature of the isothermal interior. This interior temperature \( T_m \) corresponds to the volume averaged temperature \( T_m = \langle T \rangle \).

\[
T_m = \frac{(T_b + T_s)}{2} = T_s + \Delta T/2
\]  

(10)

The model equation is derived in the following, starting from the original equation (2) in terms of the total surface heat flux. Dropping the \( \langle \cdot \rangle \) notation for internal heating we get,

\[
C \frac{d}{dt} T_m = -Q + \rho H
\]  

(11)

\[
\frac{d}{dt} T_m = -\frac{A}{CV} q_s + \frac{1}{C} \rho H
\]  

(12)

Next we express the surface heat flow density \( q_s \) in the Nusselt number with (9) and use (10) to express \( \Delta T \) in terms of \( T_m \), \( q_s = Nu k/2(T_m - T_s) \),

\[
\frac{dT_m}{dt} = -\frac{2A}{CV} Nu \frac{k}{h}(T_m - T_s) + \frac{1}{C} \rho H
\]  

(13)

From boundary layer analysis of a (quasi) stationary Rayleigh-Benard convection cell the following powerlaw relation between the Nusselt number \( Nu \) and the Rayleigh number \( Ra \) can be derived,

\[
Nu = f_N Ra^\beta
\]  

(14)

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\(^5\) See the illustrations in the lecture notes of the course in Geodynamics, *Physics of the Earth’s interior*, Fig.12, p.45. www.geo.uu.nl/ berg/geodynamics
The Rayleigh number can be expressed in the temperature contrast across the R-B layer, defined above,

\[ Ra = \frac{\rho \alpha g \Delta T h^3}{\kappa \eta}, \]  

where the parameters \( \rho, \alpha, g, h, \kappa \) and \( \eta \) are defined in Table 1 and 2.

A power law relation as in (14) between the efficiency of heat transport and the vigor of convection, expressed in the Rayleigh number, is illustrated in Fig. 1 obtained from numerical modelling of thermal convection, using numerical solutions of the coupled equations describing conductive and convective heat transport in viscous flow processes.

**problem 6** With the powerlaw relation (14), (13) represents a non-linear differential equation that can be solved numerically for the internal temperature \( T_m(t) \), for a given initial temperature \( T_m(0) \). Derive the following explicit form for this equation,

\[
\frac{dT_m}{dt} = -\frac{A}{CVh} f_N \left[ \frac{\rho \alpha g h^3}{\kappa \eta} \right]^\beta \left[ 2(T_m - T_s) \right]^\beta + 1 \frac{1}{C\rho H}
\]  

(Fig. 1: Efficiency of heat transport in Rayleigh-Benard convection obtained in series of steady state convection experiments shown by discrete symbols. Heat flux is presented by the Nusselt number as a function of the Rayleigh number. Results are shown for an isoviscous fluid and a 2-D square domain, with freeslip impermeable boundaries. A powerlaw fit in the asymptotic regime for high Rayleigh number results in a powerlaw exponent of \( \beta = 0.35 \) and prefactor \( f_N = 0.193 \).

Introducing scale parameters \( t_0 \) and \( T_0 \) for time and temperature the equation is written in terms of the non-dimensional time \( t' = t/t_0 \) and temperature \( T' = T_m/T_0 \) as,

\[
\frac{dT'}{dt'} = -2A t_0 k \frac{N u}{C V h} \left( T' - \frac{T_s}{T_0} \right) + \frac{t_0}{C T_0} \rho H = f_a N u \left( T' - \frac{T_s}{T_0} \right) + f_b H
\]  

(17)
This general form of the equation, in terms of the constant coefficients \( f_a, f_b \), is implemented in the software described in the computer lab documentation. \(^6\)

In the above model set up the boundary heat flux \( Q(t) \) is coming exclusively from conductive heat transport through the cold thermal boundary layer that is located at the outer surface of the domain. This may correspond to a domain without a bottom boundary like a full sphere. In model cases including a bottom boundary the above formulation implies that a zero heat flux condition applies there, corresponding to complete thermal insulation.

**Problem 7** The power law approximation for the Nusselt number in terms of the Rayleigh number (14), illustrated in Fig. 1 is related to a steady state Rayleigh-Benard model with anti-symmetric cold (top) and hot (bottom) thermal boundaries and an approximately isotherm, well mixed central region.

Explain how the results of the analysis for the steady state Rayleigh-Benard experiment with a heated bottom boundary can be applied to the current model set up with a thermally insulated bottom boundary. Hint: A zero heat flux condition implies \( \frac{dT}{dz} = 0 \) at the bottom boundary, where \( z \) is the depth coordinate. This suggests the use of anti-symmetric continuation of the temperature profile similar to the profile shown in Fig. 12 of the lecture notes (see footnote 5).

### 3.2 Thermal coupling of mantle and core

In thermal evolution models for terrestrial planets with a metal core the heat transport from the core into the mantle must be taken into account.

To this end the model used in computer Lab 1 characterized by a single differential equation (ODE) for the planet mantle and implicit zero heat flux bottom boundary condition is extended with a second ODE that accounts for an heat reservoir representing the planetary core. In this extended model the temperature of the core-mantle boundary (CMB) \( T_b \) is set to the temperature of the core reservoir \( T_c \),

\[
T_b(t) = T_s + \Delta T(t) = T_c(t) \tag{18}
\]

The heat flow through the CMB is driving the cooling of the core. Using the method described in section 3.1, the following ODE for the core reservoir can be derived,

\[
C_c \frac{dT_c}{dt} = - \frac{A_c}{V_c} q_c + \rho_c H_c \tag{19}
\]

The CMB heat flow density \( q_c \) can be parameterized in terms of the time dependent temperature contrast across the CMB, \( \delta T_c(t) = T_c(t) - T_m(t) \). This is done in a similar way as for the heat flow through the top boundary, where \( q_s \) was parameterized in terms of the temperature contrast across the top thermal boundary layer \( \delta T_s = T_m - T_s \), by anti-symmetric continuation over the whole mantle depth and applying a corresponding Nusselt-Rayleigh number relation, to write the heat flux in terms of the internal temperature.

Anti-symmetric continuation of the bottom thermal boundary layer over the depth of the mantle results in a temperature contrast across the mantle depth \( \Delta T_c = 2\delta T_c \). This corresponds to a Nusselt number \( Nu_c \),

\[
Nu_c = \frac{q_c}{k \Delta T_c} \Rightarrow q_c = \frac{k N u_c}{h} \Delta T_c = \frac{2k N u_c}{h} (T_c - T_m) \tag{20}
\]

\(^6\)See: [www.geo.uu.nl/~berg/planetology/Lab1.pdf](http://www.geo.uu.nl/~berg/planetology/Lab1.pdf).
In a numerical solution the Nusselt number can then be computed from the powerlaw relation in terms of the Rayleigh number with the appropriate temperature contrast $\Delta T_c = 2(T_c - T_m)$.

Substitution of $q_c$ in (19) gives,

$$\frac{dT_c}{dt} = - \left( \frac{A_c}{V_c} \right) \frac{2k}{C_c h} Nu_c(T_c - T_m) + \frac{\rho_c H_c}{C_c}$$  \hspace{1cm} (21)

Applying a time- and temperature scaling as in the Lab1 Doc., $t = t't_0$, $T = T'T_0$, with scale values $t_0$ and $T_0$ respectively we get,

$$\frac{dT'_c}{dt'} = -P_{c1} Nu_c(T'_c - T'_m) + P_{c3} H_c(t')$$  \hspace{1cm} (22)

where $P_{c1}$ and $P_{c3}$ are constant prefactors in the ODE coefficients,

$$P_{c1} = \left( \frac{A_c}{V_c} \right) \frac{2kt_0}{C_c h} , \quad P_{c3} = \frac{t_0\rho_c}{C_c T_0}$$  \hspace{1cm} (23)

The ODE for the mantle from the Lab1 model is extended now with a term corresponding to the heat flow entering the mantle through the CMB.

$$C_m \frac{dT_m}{dt} = -\frac{A_s}{V_m} q_s + A_c \frac{Nu_c}{V_m} q_c + \rho_m H_m$$  \hspace{1cm} (24)

Rewriting the Nusselt numbers in (24) with (20) and

$$Nu_s = \frac{q_s}{k\Delta T_s} $$  \hspace{1cm} $\Rightarrow q_s = \frac{kNu_s}{h} \Delta T_s = \frac{2kNu_s}{h}(T_m - T_s)$

we get the following ODE for the mantle,

$$\frac{dT'_m}{dt'} = -P_{m1} Nu_s(T'_m - T'_s) + P_{m2} Nu_c(T'_c - T'_m) + P_{m3} H_m(t')$$  \hspace{1cm} (26)

where

$$P_{m1} = \left( \frac{A_s}{V_m} \right) \frac{2kt_0}{C_m h} , \quad P_{m2} = \left( \frac{A_c}{V_m} \right) \frac{2kt_0}{C_m h} , \quad P_{m3} = \frac{t_0\rho_m}{C_m T_0}$$  \hspace{1cm} (27)

The temperature of the mantle and core are computed by solving the coupled system of ODE’s, (26), (22).  \hspace{1cm} 7

Further information about the implementation in a computerprogram for mantle-core models is given in the Lab documentation.  \hspace{1cm} 8

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7For examples of numerical solutions of coupled ODE’s see the course notes: Programming and modelling, Module 3, *Fortran programming (part 3)*, Applied to numerical solution of differential equations for planetary orbits. (www.geo.uu.nl/~berg/progmod3)

8planetology/computerlabs/Labs2017/Lab2/Doc/Lab_assignment
3.3 Temperature dependent viscosity

The viscosity of solid-state creep processes is strongly temperature dependent. This introduces a feed-back mechanism in models for a convecting mantle subject to cooling. In the early hot phase of planetary evolution the viscosity will be low and therefore the effective Rayleigh number expressed in an effective viscosity value will be high because $Ra \sim 1/\eta$. The high value of the Rayleigh number in turn corresponds to an increased vigor of convection and Nusselt number. The result is that convective cooling is faster in the initial hot stage and that it slows down during the evolution due to the negative feedback between decreasing temperature and increasing viscosity.

This effect can be included in parameterized convection models by implementing temperature dependence of viscosity in the effective Rayleigh number (15). The following parameterized viscosity relation introduced by Schubert, (1980) is often used.

$$\nu(T) = \nu_m \exp \left( \frac{A_\nu}{T} \right)$$

where $\nu$ is the so called kinematic viscosity and the usual dynamic viscosity $\eta$ is expressed as $\eta = \rho \nu$. This parameterization can be adjusted for a given pressure and temperature dependent viscosity model. Parameter values used by Schubert (1980) are listed in Table 2.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ra_{cr}$</td>
<td>critical Rayleigh number</td>
<td>1000</td>
<td>-</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>thermal expansion coefficient</td>
<td>$3 \times 10^{-5}$</td>
<td>$K^{-1}$</td>
</tr>
<tr>
<td>$g$</td>
<td>gravity acceleration</td>
<td>9.8</td>
<td>$ms^{-1}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$Nu \ Ra$ powerlaw exponent</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>$k$</td>
<td>thermal conductivity</td>
<td>5</td>
<td>$W m^{-1} k^{-1}$</td>
</tr>
<tr>
<td>$\kappa = \frac{k}{\rho c_p}$</td>
<td>thermal diffusivity</td>
<td>$7.07 \times 10^{-7}$</td>
<td>$m^2 s^{-1}$</td>
</tr>
<tr>
<td>$\eta = \rho \nu$</td>
<td>effective viscosity</td>
<td>$10^{21}$</td>
<td>$Pas$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>kinematic viscosity</td>
<td>-</td>
<td>$m^2 s^{-1}$</td>
</tr>
<tr>
<td>$\nu(T) = \nu_m \exp(\frac{A_\nu}{T})$</td>
<td>temperature dep. viscosity</td>
<td>-</td>
<td>$m^2 s^{-1}$</td>
</tr>
<tr>
<td>$\nu_m$</td>
<td>asymptotic viscosity minimum</td>
<td>$2.21 \times 10^7$</td>
<td>$m^2 s^{-1}$</td>
</tr>
<tr>
<td>$A_\nu$</td>
<td>activation temperature</td>
<td>$5.6 \times 10^4$</td>
<td>$K^{-1}$</td>
</tr>
</tbody>
</table>

Table 2: Physical parameters of parameterized convection models

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