

General conservation equations for multi-phase systems: 3. Constitutive theory for porous media flow

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Equations which describe single phase fluid flow and transport through an elastic porous media are obtained by applying constitutive theory to a set of general multiphase mass, momentum, energy, and entropy equations. Linearization of these equations yields a set of equations solvable upon specification of the material coefficients which arise. Further restriction of the flow to small velocities proves that Darcy's law is a special case of the general momentum balance.

INTRODUCTION

The description of the physical processes which occur in multi-phase systems has been a topic of practical as well as theoretical interest for many years. The early description of the behaviour of these systems was based on empirical equations and experimental observations. Typical of relationships developed in this manner is Darcy's law for flow of fluid in a porous medium which was proposed in 1856. This formula has been employed for a wide range of problems in groundwater hydrology, sanitary engineering and petroleum engineering.

The original form of Darcy's law was obtained experimentally by examining the one-dimensional flow of a single fluid through a porous column. The volumetric flow rate was found to be proportional to the head difference across the sample according to the relation:

$$Q = KA \frac{\Delta h}{L} \quad (1)$$

where K is a proportionality constant, A is the cross-sectional area of flow, Δh is the head difference, L is the thickness of the sample, and Q is the volumetric flow rate. This basic empirical relation has been arbitrarily generalized and used to describe the three-dimensional saturated and unsaturated non-isothermal flow of a compressible fluid through a deformable, moving anisotropic solid:

$$\mathbf{v}^d = -\mathbf{K} \cdot (\nabla p^f - \rho^f \mathbf{g}) \quad (2)$$

where \mathbf{v}^d is the velocity of fluid with respect to the solid, \mathbf{K} is the hydraulic conductivity tensor (assumed to be symmetric), p^f is the pore fluid pressure, ρ^f is the mass of fluid per unit volume of fluid, and \mathbf{g} is the gravity field vector.

Although the empirical extensions of Darcy's law to govern a myriad of problems have proven useful, a theoretical investigation into the thermodynamic basis for this equation will provide guidance in determining the limitations and assumptions implicit in equation (2). Additionally a general framework within which more

complex and higher order theories may be obtained will be provided. Reports presently available in the engineering literature which deal with the theoretical foundation of Darcy's law¹⁻⁴ typically consider specialized cases and have not provided a systematic and general methodology for obtaining higher order theories. Authors using the continuum theories of mixtures have modelled a porous medium as a fluid-solid mixture. Some have obtained expressions which reduce to Darcy's law under certain restrictive assumptions⁵⁻¹³.

The derivation of Darcy's law as will be presented here differs from others in the continuum mechanics literature in two fundamental ways. First, unnecessary assumptions such as fluid-phase incompressibility, constant porosity, rigid solid, constant temperature, etc., have been avoided. Second, the steps to arrive at Darcy's law starting from a general theory are carefully laid down and thus governing equations of mass, momentum, and energy for fluid-solid systems in which the Darcian constraints are too severe are obtained. These differences will be highlighted in the brief review of continuum models of porous media which follows.

The first group of continuum models of porous media developed from fundamental principles can be arbitrarily categorized as those in which the volume fraction, ϵ_f , is not explicitly considered. Some authors who use this model type^{6,8,9,14,15} include the gradient of bulk density among the independent variables. However, when this approach is followed, Darcy's law can be obtained only if the fluid phase is microscopically incompressible. When even the gradient of bulk density is not included as an independent variable (e.g. as in the constitutive equations of Bedford and Ingram¹⁰) a derivation of Darcy's law will result in the following form

$$\mathbf{v}^d = \mathbf{K} \cdot [\nabla(\epsilon_f p^f) - \epsilon_f \rho^f \mathbf{g}] \quad (3)$$

where, to highlight the difficulty with this procedure, bulk quantities have been related to intrinsic quantities through the volume fraction. Note that equation (3) incorrectly predicts the occurrence of flow due to nonuniformity of porosity.

A second category of models includes those in which the fluid volume is recognized as an important variable. Two subsets of this general class can be distinguished: one which assumes porosity is constant and one which does not. In the constant porosity case^{4,11,12,16} with $\nabla \epsilon_f \approx 0$ no inconsistencies are obtained. In some derivations where porosity is not assumed to be constant^{5,12} the final Darcy type equations have the same form as equation (3), which is incorrect as discussed earlier. Fulks *et al.*¹² propose that $\epsilon_f = \epsilon_f(p^f)$ and then using the chain rule of differentiation write:

$$\begin{aligned} \nabla p_f &= \nabla(\epsilon_f p^f) = (\epsilon_f \nabla p^f + p^f \nabla \epsilon_f) \\ &= \left(\epsilon_f \nabla p^f + p^f \frac{d\epsilon_f}{dp^f} \nabla p^f \right) = \epsilon_f \left(1 + \frac{p^f}{\epsilon_f} \frac{d\epsilon_f}{dp^f} \right) \nabla p^f \quad (4) \end{aligned}$$

Fulks *et al.* next propose that in some instances:

$$\frac{p^f}{\epsilon_f} \frac{d\epsilon_f}{dp^f} \ll 1 \quad (5)$$

and the usual form of Darcy's law may be recovered. This result is not general because even for cases where equation (5) does not hold, Darcy's law will be given by equation (2).

In the work of Garg¹⁷ and Garg *et al.*¹⁸ a 'momentum equation for fluid flow through variable cross-section' is postulated such that by comparison with the usual fluid-phase momentum balance the constitutive equation needed to obtain Darcy's law may be discerned. However, such an approach is artificial because the field equations should be obtained only after the constitutive relations are found.

Some authors have employed an averaging approach to derive appropriate expressions for fluid flow in a solid matrix^{1,19}. By postulating linear constitutive equations Saffman²⁰, Kenyon⁷, and Gray and O'Neill¹ have obtained Darcy's law. Bedford and Drumheller²¹ have derived a general momentum equation without the viscous stress term through application of Hamilton's extended variational principle. In all of these works the question which remains unanswered is how to obtain higher order theories. Also many of the procedures adopted do not apply when more than one fluid is present.

In providing a systematic procedure for deriving multiphase field equations, Hassanizadeh and Gray^{22,23} have applied the technique of local averaging to the microscopic equations which govern the flow and transport processes occurring in each phase. This procedure conceptually changes the description of a multiphase system from a number of juxtaposed phases to a system of overlapping continua existing simultaneously everywhere with each phase occupying only a fraction of each element of volume. To be truly useful for the description of multiphase systems, the multiphase equations of Hassanizadeh and Gray must be supplemented by some constitutive relations.

In the present work, a general set of constitutive equations is postulated and the Coleman and Noll²⁴ method of exploitation of the entropy inequality is employed to obtain proper forms of the constitutive equations which do not violate the second law of thermodynamics. The constitutive equations are then linearized and the conditions under which the linearized equations

of motion for the fluid phase reduce to Darcy's law are exposed.

KINEMATIC EQUATIONS AND BALANCE LAWS

Kinematics

As discussed earlier, a porous medium can be viewed as a body which, in the case of single-phase flow, consists of two coexistent continua, one solid and one fluid. Each continuum possesses a reference configuration at time $t = 0$ which will be altered by its motion. The two motions are independent. In particular, a typical particle of the solid phase occupying a reference position X_k^s at time $t = 0$, is carried to the spatial point x_k by the motion. The solid phase motion is defined by a function $F_k^s(X_k^s, t)$ such that:

$$x_k = F_k^s(X_k^s, t) \quad k, K = 1, 2, 3 \quad (6)$$

Throughout the text capital indices are used to refer to material coordinates while lower case indices refer to spatial coordinates.

According to the axiom of continuity of matter²⁵ there should exist a single-valued inverse of (6) such that:

$$X_k^s = F_k^{s-1}(x_k, t) \quad k, K = 1, 2, 3 \quad (7)$$

where F_k^{s-1} defines the solid phase inverse motion. For F_k^{s-1} to be single-valued, the Jacobian of the motion must be positive in the neighbourhood of X_k^s ²⁵ i.e.,

$$J^s = \det F_{kK}^s > 0 \quad (8)$$

where

$$F_{kK}^s = \partial x_k / \partial X_K^s \quad (9)$$

is the solid phase deformation gradient. The measure of solid deformation employed herein will be the Lagrangian strain tensor given as:

$$E_{KL}^s = \frac{1}{2}(F_{kK}^s F_{kL}^s - \delta_{KL}) \quad (10)$$

where δ_{KL} is the Kronecker delta.

The material time rate of change, with respect to the solid phase, of a tensorial quantity $\mathbf{f} = \mathbf{f}(X^s, t)$ is defined as:

$$\frac{D\mathbf{f}}{Dt} = \frac{\partial \mathbf{f}(X_k^s, t)}{\partial t} \Big|_{X_k^s} \quad (11)$$

where $\Big|_{X_k^s}$ indicates that X_k^s is held constant. The solid phase velocity is the material time rate of change of its motion, i.e.

$$v_k^s = \frac{D}{Dt} F_k^s(X_k^s, t) = \frac{\partial F_k^s}{\partial t} \quad (12)$$

where $v_k^s = v_k^s(\mathbf{x}, t)$ is the solid phase velocity field. By use of

equations (8), (9), (12) and the chain rule of differentiation it can be proven that:

$$\frac{\overset{s}{D}J^s}{Dt} = J^s v_{k,k}^s \quad (13)$$

If the tensorial quantity \mathbf{f} is a function of the spatial position x_k and time rather than X_K^s and time, then, by use of the chain rule of differentiation, one finds that

$$\frac{\overset{s}{D}}{Dt} \mathbf{f}(x_k, t) = \frac{\partial \mathbf{f}}{\partial t} + v_k^s \frac{\partial \mathbf{f}}{\partial x_k} \quad (14a)$$

A similar relation will give the material time rate of change of \mathbf{f} , with respect to the fluid phase:

$$\frac{\overset{f}{D}}{Dt} \mathbf{f}(x_k, t) = \frac{\partial \mathbf{f}}{\partial t} + v_k^f \frac{\partial \mathbf{f}}{\partial x_k} \quad (14b)$$

where $v_k^f(x_k, t)$ is the fluid velocity field.

Subtraction of equation (14a) from (14b) yields the following relation:

$$\frac{\overset{f}{D}}{Dt} \mathbf{f}(x_k, t) = \frac{\overset{s}{D}\mathbf{f}}{Dt} + v_k^d \frac{\partial \mathbf{f}}{\partial x_k} \quad (15)$$

where

$$v_k^d = v_k^f - v_k^s \quad (16)$$

called the diffusion velocity, is the velocity of the fluid with respect to the solid. Also the mixture velocity denoted by \mathbf{v} , may be defined as:

$$\rho \mathbf{v}_k = \rho_f v_k^f + \rho_s v_k^s \quad (17)$$

where ρ_f is the mass of fluid per unit volume of mixture, ρ_s is the mass of solid per unit volume of mixture, and $\rho = \rho_f + \rho_s$ is the mass density of the mixture. Thus a material time derivative with respect to the mixture velocity can be defined as:

$$\frac{D\mathbf{f}}{Dt} = \frac{\partial \mathbf{f}}{\partial t} + v_k \frac{\partial \mathbf{f}}{\partial x_k} \quad (18)$$

It can be easily proven that:

$$\frac{D\mathbf{f}}{Dt} = \frac{\overset{\alpha}{D}\mathbf{f}}{Dt} - v_k^{\alpha} \frac{\partial \mathbf{f}}{\partial x_k} \quad \alpha = f, s \quad (19a)$$

where v_k^{α} is the velocity of the α -phase with respect to the velocity of the mixture, i.e.

$$v_k^{\alpha} = v_k^{\alpha} - v_k \quad (19b)$$

The material time derivative of E_{kL}^s , which will be needed later, is found to be²⁵

$$\frac{\overset{s}{D}E_{kL}^s}{Dt} = d_{kl}^s F_{(kK}^s F_{lL}^s \quad (20a)$$

where equations (9)–(12) have been used and

$$d_{kl}^{\alpha} = \frac{1}{2}(v_{k,l}^{\alpha} + v_{l,k}^{\alpha}) \quad \alpha = f, s \quad (20b)$$

is the deformation rate tensor of the α -phase, and $F_{(kK}^s F_{lL}^s$ denotes the symmetric part of $F_{kK}^s F_{lL}^s$, i.e.

$$F_{(kK}^s F_{lL}^s = \frac{1}{2}(F_{kK}^s F_{lL}^s + F_{lK}^s F_{kL}^s) \quad (20c)$$

Balance laws

Developing and utilizing a systematic averaging technique, Hassanizadeh and Gray^{22,23} derived the equations of balance for a multi-phase system from the classical equations of continuum mechanics. Their results for the α th phase are

Conservation of mass

$$\frac{\overset{\alpha}{D}\langle \rho \rangle_{\alpha}}{Dt} + \langle \rho \rangle_{\alpha} \bar{v}_{k,k}^{\alpha} = \langle \rho \rangle_{\alpha} e^{\alpha}(\rho) \quad (21a)$$

Conservation of momentum

$$\langle \rho \rangle_{\alpha} \frac{\overset{\alpha}{D}\bar{v}_i^{\alpha}}{Dt} - t_{ki,k}^{\alpha} - \langle \rho \rangle_{\alpha} \bar{g}_i^{\alpha} - \langle \rho \rangle_{\alpha} \hat{T}_i^{\alpha} - \langle \rho \rangle_{\alpha} e^{\alpha}(\rho \bar{v}_i) = 0 \quad (21b)$$

Angular momentum balance

$$t_{ki}^{\alpha} = t_{ik}^{\alpha} \quad (21c)$$

Conservation of energy

$$\langle \rho \rangle_{\alpha} \frac{\overset{\alpha}{D}E^{\alpha}}{Dt} - t_{ki}^{\alpha} \bar{v}_i^{\alpha} - q_{k,k}^{\alpha} - \langle \rho \rangle_{\alpha} h^{\alpha} - \langle \rho \rangle_{\alpha} \hat{Q}^{\alpha} - \langle \rho \rangle_{\alpha} (e^{\alpha}(\rho \hat{E}) - e^{\alpha}(\rho) E^{\alpha}) = 0 \quad (21d)$$

Entropy balance

$$\langle \rho \rangle_{\alpha} \frac{\overset{\alpha}{D}\bar{S}^{\alpha}}{Dt} - \varphi_{k,k}^{\alpha} - \langle \rho \rangle_{\alpha} \frac{h^{\alpha}}{\theta^{\alpha}} - \langle \rho \rangle_{\alpha} \hat{\Phi}^{\alpha} - \langle \rho \rangle_{\alpha} (e^{\alpha}(\rho S) - e^{\alpha}(\rho) \bar{S}^{\alpha}) = \langle \rho \rangle_{\alpha} \Gamma^{\alpha} \quad \alpha = f, s \quad (21e)$$

subject to

$$\sum_{\alpha=f,s} \langle \rho \rangle_{\alpha} e^{\alpha}(\rho) = 0 \quad (22a)$$

$$\sum_{\alpha=f,s} \langle \rho \rangle_{\alpha} (e^{\alpha}(\rho) \bar{v}_i^{\alpha} + e^{\alpha}(\rho \bar{v}_i) + \hat{T}_i^{\alpha}) = 0 \quad (22b)$$

$$\varepsilon_f + \varepsilon_s = 1 \quad (24b)$$

$$\sum_{\alpha=f,s} \langle \rho \rangle_{\alpha} \left(\frac{1}{2} e^{\alpha}(\rho) \bar{v}_i^{\alpha} \bar{v}_i^{\alpha} + e^{\alpha}(\rho \bar{v}_i) \bar{v}_i^{\alpha} + e^{\alpha}(\rho \hat{E}) + \hat{T}_i^{\alpha} \bar{v}_i^{\alpha} + \hat{Q}^{\alpha} \right) = 0 \quad (22c)$$

$$\sum_{\alpha=f,s} \langle \rho \rangle_{\alpha} (e^{\alpha}(\rho S) + \hat{\Phi}^{\alpha}) = 0 \quad (22d)$$

where $\langle \rho \rangle_{\alpha}$ is the density of the α -phase, the mass of α per total volume of the system, $e^{\alpha}(\rho)$ is the exchange of mass between the α -phase and all other phases, \bar{v}_i^{α} is the intrinsic α -phase mass average velocity, t_{ki}^{α} is the α -phase stress tensor, \bar{g}_i^{α} is the α -phase external supply of momentum, \hat{T}_i^{α} is the exchange of momentum between the α -phase and all other phases, due to mechanical interaction, $e^{\alpha}(\rho \bar{v}_i)$ is the exchange of momentum between the α -phase and all other phases due to exchange of mass, E^{α} is the α -phase macroscopic internal energy density function, q_k^{α} is the α -phase heat flux, h^{α} is the α -phase external supply of energy, \hat{Q}^{α} is the exchange of internal energy between the α -phase and all other phases, due to mechanical interactions, $e^{\alpha}(\rho \hat{E})$ is the exchange of energy between the α -phase and all other phases, due to exchange of mass, S^{α} is the α -phase internal entropy density function, φ_i^{α} is the α -phase entropy flux, θ^{α} is the α -phase absolute temperature function, $\hat{\Phi}^{\alpha}$ is the exchange of entropy between the α -phase and all other phases, due to mechanical interactions, $e^{\alpha}(\rho S)$ is the exchange of entropy between the α -phase and all other phases, due to exchange of mass, Γ^{α} is the α -phase net production of entropy, and $\sum_{\alpha=f,s}$ indicates summation over the fluid and solid phases. The angular brackets and overbars are included in these equations to enable the reader to establish contact with the previous work of Hassanizadeh and Gray. However, in the interest of simplicity, these averaging symbols will be omitted in the remainder of the present work.

The scope of problems to be considered is restricted to the type of porous media in which no mass exchange phenomena (e.g. phase change, adsorption) is occurring between the fluid and solid phases. Thus the following quantities will be identically zero.

$$e^{\alpha}(\rho) = e^{\alpha}(\rho \bar{v}) = e^{\alpha}(\rho \hat{E}) = e^{\alpha}(\rho S) \equiv 0 \quad \alpha = f, s \quad (23a)$$

Furthermore, the entropy flux is assumed proportional to the heat flux such that:

$$\varphi_k^{\alpha} = \frac{q_k^{\alpha}}{\theta^{\alpha}} \quad (23b)$$

Obviously, equation (23b) serves as a constitutive assumption for the entropy flux. Alternatively, one may leave φ_k^{α} unspecified and later postulate a constitutive equation for it²⁶. For a discussion of these two alternatives see Green and Naghdi²⁷.

Finally, the mass density functions, ρ_{α} , are replaced by:

$$\rho_{\alpha} = \varepsilon_{\alpha} \rho^{\alpha} \quad (24a)$$

where ρ^{α} and ε_{α} are respectively the intrinsic mass density and the volume fraction of the α th phase. For the fluid-solid system considered here, one has

Thus, substitution of equations (23) and (24) into (21) and (22) yields the following balance equations.

$$\frac{D \varepsilon_{\alpha} \rho^{\alpha}}{Dt} + \varepsilon_{\alpha} \rho^{\alpha} v_{k,k}^{\alpha} = 0 \quad (25a)$$

$$\varepsilon_{\alpha} \rho^{\alpha} \frac{D v_i^{\alpha}}{Dt} - t_{ki,k}^{\alpha} - \varepsilon_{\alpha} \rho^{\alpha} g_i^{\alpha} - \varepsilon_{\alpha} \rho^{\alpha} \hat{T}_i^{\alpha} = 0 \quad (25b)$$

$$t_{ki}^{\alpha} = t_{ik}^{\alpha} \quad (25c)$$

$$\varepsilon_{\alpha} \rho^{\alpha} \frac{D E^{\alpha}}{Dt} - t_{ki}^{\alpha} v_{i,k}^{\alpha} - q_{k,k}^{\alpha} - \varepsilon_{\alpha} \rho^{\alpha} h^{\alpha} - \varepsilon_{\alpha} \rho^{\alpha} \hat{Q}^{\alpha} = 0 \quad (25d)$$

$$\varepsilon_{\alpha} \rho^{\alpha} \Gamma^{\alpha} = \varepsilon_{\alpha} \rho^{\alpha} \frac{D S^{\alpha}}{Dt} - (q_k^{\alpha} / \theta^{\alpha})_{,k} - \varepsilon_{\alpha} \rho^{\alpha} h^{\alpha} / \theta^{\alpha} - \varepsilon_{\alpha} \rho^{\alpha} \hat{\Phi}^{\alpha} \quad (25e)$$

$$\alpha = f, s$$

subject to:

$$\varepsilon_f \rho^f \hat{T}_i^f + \varepsilon_s \rho^s \hat{T}_i^s = 0 \quad (26a)$$

$$\varepsilon_f \rho^f (\hat{T}_i^f v_i^f + \hat{Q}^f) + \varepsilon_s \rho^s (\hat{T}_i^s v_i^s + \hat{Q}^s) = 0 \quad (26b)$$

$$\varepsilon_f \rho^f \hat{\Phi}^f + \varepsilon_s \rho^s \hat{\Phi}^s = 0 \quad (26c)$$

Second law of thermodynamics

The second law of thermodynamics dictates the sign of net entropy production. According to this law, the net entropy production rate of the system is always positive. Hassanizadeh and Gray²³ have derived the following form of the second law:

$$\Gamma = \sum_{\alpha=f,s} \varepsilon_{\alpha} \rho^{\alpha} \Gamma^{\alpha} \geq 0 \quad (27)$$

Elimination of h^{α} between (25d) and (25e), and substitution of the result into (27) yields:

$$\Gamma = \sum_{\alpha=f,s} \frac{1}{\theta^{\alpha}} \left\{ -\varepsilon_{\alpha} \rho^{\alpha} \left(\frac{D A^{\alpha}}{Dt} + S^{\alpha} \frac{D \theta^{\alpha}}{Dt} \right) + t_{ki}^{\alpha} v_{ki}^{\alpha} + \frac{1}{\theta^{\alpha}} q_k^{\alpha} \theta_{,k}^{\alpha} \right\} - \frac{1}{\theta^s} \varepsilon_f \rho^f \hat{T}_k^f v_k^f - \frac{1}{\theta^f \theta^s} (\theta^f - \theta^s) \varepsilon_f \rho^f \hat{Q}^f \geq 0 \quad (28)$$

where equations (20b), (25c), (16), and (26) have been used, and A^{α} is the Helmholtz free energy of the α -phase defined as:

$$A^{\alpha} = E^{\alpha} - \theta^{\alpha} S^{\alpha} \quad (29)$$

Finally, attention will be confined to the case where both solid and fluid have the same temperature θ , i.e.

$$\theta^f = \theta^s = \theta \quad (30)$$

As a result, energy equations for each phase need not be considered. Rather a total energy equation obtained as the sum of fluid- and solid-phase energy equations will be used:

$$\rho \frac{DE}{Dt} - \sum_{\alpha} [t_{kl}^{\alpha} v_{l,k}^{\alpha} + q_{k,k}^{\alpha} - (\epsilon_{\alpha} \rho^{\alpha} E^{\alpha} v_k^{\alpha})_{,k}] - \rho h + \epsilon_f \rho^f \hat{T}_k^f v_k^d = 0 \quad (31)$$

where equations (26a) and (26b) have been used to express \hat{Q}^{α} in terms of $\hat{T}_k^{\alpha} v_k^d$, and E and h are the internal energy and external supply of energy for the whole body, respectively, such that:

$$\rho E = \sum_{\alpha=f,s} \rho_{\alpha} E^{\alpha} = \sum_{\alpha=f,s} \epsilon_{\alpha} \rho^{\alpha} E^{\alpha} \quad (32)$$

and

$$\rho h = \sum_{\alpha=f,s} \rho_{\alpha} h^{\alpha} = \sum_{\alpha=f,s} \epsilon_{\alpha} \rho^{\alpha} h^{\alpha} \quad (33)$$

Also, equation (28) will become:

$$\theta \Gamma = \sum_{\alpha=f,s} \left\{ -\epsilon_{\alpha} \rho^{\alpha} \left(\frac{D A^{\alpha}}{Dt} + S^{\alpha} \frac{D \theta}{Dt} \right) + t_{kl}^{\alpha} d_{kl}^{\alpha} + \left(\frac{q_k^{\alpha}}{\theta} - \epsilon_{\alpha} \rho^{\alpha} S^{\alpha} v_k^{\alpha} \right) \theta_{,k} \right\} - \epsilon_f \rho^f \hat{T}_k^f v_k^d \geq 0 \quad (34)$$

where equations (19a) and (30) have been invoked.

Multiplication of equation (29) by $\epsilon_{\alpha} \rho^{\alpha}$ and summation over the fluid and solid phases yield:

$$A = E - \theta S \quad (35)$$

where A and S are the Helmholtz free energy and the intrinsic entropy of the whole body, defined respectively by:

$$\rho A = \sum_{\alpha=f,s} \rho_{\alpha} A^{\alpha} = \sum_{\alpha=f,s} \epsilon_{\alpha} \rho^{\alpha} A^{\alpha} \quad (36a)$$

and

$$\rho S = \sum_{\alpha=f,s} \rho_{\alpha} S^{\alpha} = \sum_{\alpha=f,s} \epsilon_{\alpha} \rho^{\alpha} S^{\alpha} \quad (36b)$$

CONSTITUTIVE EQUATIONS

The balance laws given as equations (25a), (25b), and (31), and inequality (34) constitute nine equations in 35 unknowns, namely:

$$\epsilon_f, \rho^f, \rho^s, v_k^{\alpha}, \theta, t_{kl}^{\alpha}, \hat{T}_k^{\alpha}, E^{\alpha} \text{ (or } A^{\alpha}), q_k^{\alpha}, S^{\alpha} \quad \alpha=f, s; k, l=1, 2, 3 \quad (37)$$

(Because restrictions (24b) and (26a) may be used to determine ϵ_s and \hat{T}_k^s in terms of the corresponding fluid-phase properties, these quantities need not be treated as unknowns.) Hence 26 additional equations are needed in order to have a determinate system. Therefore constitutive equations must be proposed which account for the material properties of the system under consideration, expose the inter-relation of field properties, and remove the deficit in the number of equations. These equations will be obtained by selecting 25 dependent variables and imposing a constraint on the type of solid phase considered.

The 25 dependent variables are chosen to be:

$$t_{kl}^{\alpha}, \hat{T}_k^{\alpha}, A^{\alpha}, q_k^{\alpha}, S^{\alpha} \quad \alpha=f, s; k, l=1, 2, 3 \quad (38)$$

These variables are not directly measurable but will be determined as functions of directly measurable (independent) variables. The choice of independent variables depends on the types of solids and fluids, as well as the types of phenomena, to be encountered. In this work an elastic, microscopically incompressible solid and a viscous fluid capable of heat conduction and diffusion are considered. A practical case for which the solid phase may be considered microscopically compressible is when the solid phase is composed of rigid or incompressible grains. In such a case, ρ^s will remain constant. Note that although the solid grains may be incompressible, the porous matrix itself may deform and compress. Therefore the independent variables chosen are: ρ^f , the fluid density, $E_{K_L}^s$, the solid phase strain tensor, which accounts for the elasticity of the solid, ϵ_f , $\epsilon_{f,k}$, the porosity and porosity gradient, which account for local volume changes and buoyancy effects, d_{kl}^f , the fluid-phase deformation rate tensor, which accounts for the fluid viscosity, $\theta, \theta_{,k}$, the temperature and temperature gradient, so that heat conduction may be included, v_k^d , the relative velocity of the fluid with respect to the solid. Because the solid is considered to be microscopically incompressible, the solid phase mass density must satisfy the restriction that

$$\frac{D \rho^s}{Dt} = 0 \quad (39a)$$

or upon integration

$$\rho^s = \rho_0^s(X^s) \quad (39b)$$

where ρ_0^s is the initial mass density of the solid phase. Substitution of equations (39a) and (13) into the solid-phase continuity equation (25a) and rearrangement yield:

$$\frac{D \epsilon_s J^s}{Dt} = 0 \quad (40a)$$

or upon integration:

$$\epsilon_s = \epsilon_{s0}(X^s)/J^s \quad (40b)$$

where ϵ_{s0} is the initial value of the solid-phase volume fraction. Because J^s can be expressed in terms of the invariants of the strain tensor $E_{K_L}^s$,²⁵ knowledge of $E_{K_L}^s$ determines ϵ_s through the continuity equation (40). Thus,

one need not explicitly include ε_f in the list of independent variables. Later, however, when the constitutive equations are approximated by linear expressions, the explicit dependence on ε_f will be resumed.

Now, the constitutive equations may be postulated in the following form:

$$A^s = A^s(\rho^f, \varepsilon_{f,k}, E_{Klb}^s d_{kl}^f, \theta, \theta_{,k}, v_k^d) \quad (41a)$$

$$S^s = S^s(\rho^f, \varepsilon_{f,k}, E_{Klb}^s d_{kl}^f, \theta, \theta_{,k}, v_k^d) \quad (41b)$$

$$t_{kl}^s = t_{kl}^s(\rho^f, \varepsilon_{f,k}, E_{Klb}^s d_{kl}^f, \theta, \theta_{,k}, v_k^d) \quad (41c)$$

$$q_k^s = q_k^s(\rho^f, \varepsilon_{f,k}, E_{Klb}^s d_{kl}^f, \theta, \theta_{,k}, v_k^d) \quad (41d)$$

$$\hat{T}_k^f = \hat{T}_k^f(\rho^f, \varepsilon_{f,k}, E_{Klb}^s d_{kl}^f, \theta, \theta_{,k}, v_k^d) \quad (41e)$$

These equations are restricted by principles of objectivity and admissibility. According to the principle of objectivity, the constitutive equations shall remain unchanged under an orthonormal transformation of the spatial frame of reference. The principle of admissibility states that the constitutive equations shall not violate the balance laws and the second law of thermodynamics. The last requirement reveals much information on the constitutive functions and will be fully studied.

The Coleman and Noll²⁴ method for exploitation of the entropy inequality is used to obtain the restrictions imposed on the constitutive equations by the second law of thermodynamics. The details of this procedure are presented in Appendix A and lead to the following simplifications of the constitutive equations (41):

$$A^f = A^f(\rho^f, \theta) \quad (42)$$

$$A^s = A^s(E_{Klb}^s, \theta) \quad (43)$$

$$A = A(\rho^f, E_{Klb}^s, \theta) \quad (44)$$

$$S = S(\rho^f, E_{Klb}^s, \theta) = -\frac{\partial A}{\partial \theta} \quad (45)$$

$$E = E(\rho^f, E_{Klb}^s, \theta) = A + \theta S \quad (46)$$

$$t_{kl}^s = -\varepsilon_s(\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \delta_{kl} + \varepsilon_s \rho_s^0 \frac{\partial A^s}{\partial E_{Klb}^s} F_{(kl}^s F_{nl}^s \quad (47)$$

The quantities S^s , t_{kl}^f , q_k^s , and \hat{T}_k^f retain the complete dependence on the independent variables given by equations (41b)–(41e) but this dependence is subjected to the residual entropy inequality,

$$\begin{aligned} \theta \Gamma = & \left[\varepsilon_f(\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \delta_{kl} + t_{kl}^f \right] d_{kl}^f + \left[\frac{1}{\theta} (q_k^f + q_k^s) - \right. \\ & \left. \varepsilon_f \rho^f \left(S^f + \frac{\partial A^f}{\partial \theta} \right) v_k^d \right] \theta_{,k} - \\ & \left[-(\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \varepsilon_{f,k} + \varepsilon_f \rho^f \hat{T}_k^f \right] v_k^d \geq 0 \quad (48) \end{aligned}$$

This residual inequality can be used to extract additional information by examining the behaviour of the porous medium system at equilibrium.

Equilibrium properties

The state of thermodynamic equilibrium is defined to be the state at which the following independent variables are all zero:

$$\{Z_A; A=1, 12\} = \{d_{kl}^f, \theta_{,k}, v_k^d\} \quad (49)$$

From equation (48) it is readily verified that Γ vanishes at equilibrium, and hence, it is a minimum at $Z_A=0$, $A=1, 12$. The necessary and sufficient conditions for Γ to be minimum at equilibrium are:

$$\left. \frac{\partial \Gamma}{\partial Z_A} \right|_e = 0 \quad A=1, 12 \quad (50a)$$

$$\left\| \left. \frac{\partial^2 \Gamma}{\partial Z_A \partial Z_B} \right|_e \right\| \text{ is positive definite } A, B=1, 12 \quad (50b)$$

where $|_e$ indicates that the function preceding the bar is evaluated at equilibrium. These conditions place restrictions on the constitutive functions. In particular imposition of equation (50a) on (48) yields:

$$(q_k^f + q_k^s)|_e = 0 \quad (51)$$

$$t_{kl}^f|_e = -\varepsilon_f \rho^f \delta_{kl} \quad (52)$$

$$\varepsilon_f \rho^f \hat{T}_k^f|_e = p^f \varepsilon_{f,k} \quad (53)$$

where

$$p^f = p^f(\rho^f, \theta) = (\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \quad (54)$$

is the equilibrium value of the fluid-pressure stress tensor and is called 'thermodynamic pressure'. The appearance of ε_f in equation (52) multiplying p^f is reasonable because t_{kl}^f is the force exerted on the fluid-phase per unit area of the mixture.

General equations

Exploitation of the entropy inequality and consideration of the equilibrium characteristics of the liquid-solid system has led to considerable simplification of the constitutive equations (41). Additional manipulation in Appendix B aimed towards representing DE/Dt in terms of D/Dt of independent variables also reveals that S^s is equal to the negative of the derivative of A^s with respect to θ . Thus the constitutive equations are of the form

$$A^f = A^f(\rho^f, \theta) \quad (55a)$$

$$A^s = A^s(E_{Klb}^s, \theta) \quad (55b)$$

$$S^f = S^f(\rho^f, \theta) = -\frac{\partial A^f}{\partial \theta} \quad (56a)$$

$$S^s = S^s(E_{Klb}^s, \theta) = -\frac{\partial A^s}{\partial \theta} \quad (56b)$$

$$E^f = E^f(\rho^f, \theta) = A^f - \theta \frac{\partial A^f}{\partial \theta} \quad (57a)$$

$$E^s = E^s(E_{Klb}^s, \theta) = A^s - \theta \frac{\partial A^s}{\partial \theta} \quad (57b)$$

$$t_{ki}^f = -\varepsilon_f p^f \delta_{ki} + \tau_{ki}^f(\rho^f, \varepsilon_{f,k}, E_{Kb}^f, d_{ki}^f, \theta, \theta_{,k}, v_k^f) \quad (58a)$$

$$t_{ki}^s = -\varepsilon_s p^s \delta_{ki} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial E_{KL}^s} F_{(kK}^s F_{iL}^s \quad (58b)$$

$$q_k^s = q_k^s(\rho^s, \varepsilon_{f,k}, E_{Kb}^s, d_{ki}^s, \theta, \theta_{,k}, v_k^s) \quad (59)$$

$$\varepsilon_f \rho^f \hat{T}_k^f = p^f \varepsilon_{f,k} + \hat{\tau}_k^f(\rho^f, \varepsilon_{f,k}, E_{Kb}^f, d_{ki}^f, \theta, \theta_{,k}, v_k^f) \quad (60)$$

where τ_{ki}^f and $\hat{\tau}_k^f$ are the dissipative parts of the fluid-phase stress tensor and momentum exchange term, respectively, and because of equations (51)–(53):

$$\tau_{ki}^f(\rho^f, \varepsilon_{f,k}, E_{Kb}^f, 0, \theta, 0, 0) = 0 \quad (61)$$

$$q_k^f(\rho^f, \varepsilon_{f,k}, E_{Kb}^f, 0, \theta, 0, 0) + q_k^s(\rho^s, \varepsilon_{f,k}, E_{Kb}^s, 0, \theta, 0, 0) = 0 \quad (62)$$

$$\hat{\tau}_k^f(\rho^f, \varepsilon_{f,k}, E_{Kb}^f, 0, \theta, 0, 0) = 0 \quad (63)$$

Substitution of these constitutive forms into equation (48) and use of the identities (54), and (B24) yields the entropy inequality in the form:

$$\theta \Gamma = \tau_{ki}^f d_{ki}^f + \frac{1}{\theta} [q_k^f + q_k^s] \theta_{,k} - \hat{\tau}_k^f v_k^f \geq 0 \quad (64)$$

Combination of these constitutive forms with the general balance equations (25) yields the following field equations for the fluid as well as the solid phase.

Conservation of fluid mass

$$\frac{\partial}{\partial t} (\varepsilon_f \rho^f) + (\varepsilon_f \rho^f v_k^f)_{,k} = 0 \quad (65)$$

Conservation of fluid momentum

$$\varepsilon_f \rho^f \frac{\partial v_k^f}{\partial t} + \varepsilon_f \rho^f v_k^f v_{k,i}^f = -\varepsilon_f p_{,k}^f + \tau_{ki,i}^f + \varepsilon_f \rho^f g_k^f + \hat{\tau}_k^f \quad (66)$$

Conservation of solid mass

$$-\frac{\partial \varepsilon_f}{\partial t} + [(1 - \varepsilon_f) v_k^s]_{,k} = 0 \quad (67)$$

Conservation of solid momentum

$$(1 - \varepsilon_f) \rho_0^s \frac{\partial^2 F_k^s}{\partial t^2} = (\varepsilon_f - 1) p_{,k}^f + \left[(1 - \varepsilon_f) \rho_0^s \frac{\partial A^s}{\partial E_{KL}^s} F_{(kK}^s F_{iL}^s \right]_{,i} + (1 - \varepsilon_f) \rho_0^s g_k^s - \hat{\tau}_k^f \quad (68)$$

The energy equation for the whole body is also obtained by substitution of the constitutive forms of this section into equation (31). The term DE/Dt has been expanded in Appendix B. Hence:

$$\frac{\varepsilon_f}{\rho^f} \left[(1 - \varepsilon_f) \frac{\rho^f \rho_0^s}{\rho} (E^f - E^s) + \left(p^f - \theta \frac{\partial p^f}{\partial \theta} \right) \right] \frac{D\rho^f}{Dt} +$$

$$\left[(1 - \varepsilon_f) \frac{\rho^f \rho_0^s}{\rho} (E^f - E^s) B_{KL}^s + (1 - \varepsilon_f) \rho_0^s \frac{\partial E^s}{\partial E_{KL}^s} \right] \frac{DE_{KL}^s}{Dt} +$$

$$\rho C_v \frac{D\theta}{Dt} = -p^f (\varepsilon_f v_k^f)_{,k} - p^s v_{k,k}^s + \tau_{ki}^f d_{ki}^f +$$

$$(1 - \varepsilon_f) \rho_0^s \frac{\partial A^s}{\partial E_{KL}^s} F_{(kK}^s F_{iL}^s d_{ki}^s + (q_k^f + q_k^s)_{,k} -$$

$$\left[\varepsilon_f (1 - \varepsilon_f) \frac{\rho^f \rho_0^s}{\rho} (E^f - E^s) v_k^f \right]_{,k} - \hat{\tau}_k^f v_k^f + \rho h \quad (69)$$

where the term

$$\sum_a \varepsilon_a \rho^a E^a v_k^a$$

has been rearranged to the form

$$\frac{\rho_f \rho_s}{\rho} (E^f - E^s) v_k^f$$

and B_{KL}^s is defined in equation (B15). The first term in the l.h.s. of this equation is non-zero when the fluid is compressible. It consists of two contributions to the energy balance. The first part is the dispersion of internal energy when the fluid compresses and the fluid and solid internal energies are not equal. The second part is the change of internal energy due to the work of pressure on the fluid-phase volume change. The next term in the l.h.s. of equation (69) represents the change of energy due to the deformability of the solid matrix. Similar to the first term, it also consists of two parts. The last term on the l.h.s. of equation (69) represents the internal energy change due to change in temperature and C_v is the heat capacity. The first four terms on the r.h.s. of equation (69) account for the work of the fluid-phase pressure and viscous stress and the solid-phase stress due to the fluid- and solid-phase motions. The fifth term represents the heat flux, and the sixth term accounts for the dispersion of energy due to the relative motion of the fluid with respect to the solid. The seventh term is the energy dissipation term due to interfacial drag between the fluid- and solid-phases. Finally, the last term is the external supply of energy to the system (e.g. radiation).

The constitutive functions τ_{ki}^f , $\hat{\tau}_k^f$, A^f , A^s , q_k^f and q_k^s (or in fact $q_k^f + q_k^s$) which appear in the field equations (65)–(69), must be determined experimentally. Therefore, the field equations provide nine equations in terms of nine unknowns, which are ρ^f , ε_f , v_k^f , F_k^s , and θ . Note that, through kinematic relations, quantities such as v_k^s , E_{KL}^s and d_{ki}^s are expressible in terms of these unknowns. Because determination of the constitutive functions purely by experiment is a formidable task, polynomial expansions are often employed up to the desired degree of approximation. In the next section a linear theory will be developed.

LINEAR THEORY

In many engineering problems, attainment of a high degree of generality is not helpful and, very often, not practical. In fact, linear theories are the ones most widely employed, and, for a large class of problems, yield satisfactory results. Hence in this section a theory that will be linear in the following variables is developed:

$$\{Z_A, A=1, 21\} = \{\varepsilon_{f,k}, e_{ki}^f, d_{ki}^f, \theta_{,k}, v_k^f\} \quad (70)$$

where e_{ki}^f is the infinitesimal strain tensor approximating E_{KL}^f in the linear theory²⁵.

The constitutive functions (55)–(60), restricted by equations (61)–(64), are linearized by expanding them in a Taylor series around $Z_A=0, A=1, 21$. Then quadratic and higher order terms are neglected except in the case of the solid-phase free energy function where quadratic terms are also kept since it serves as a potential function for the stress tensor. The coefficients of these series, which account for the material properties, are called material coefficients.

The linearization procedure, including the investigation of the consequences of isotropy requirements and restrictions imposed by the second law, is lengthy and is given in Appendix C. The final forms of the linearized conservation equations after substitution of the linearized constitutive relations are given below where the material coefficients which appear have been defined in Appendix C.

Conservation of fluid mass

$$\frac{\partial}{\partial t}(\varepsilon_f \rho^f) + (\varepsilon_f \rho^f v_k^f)_{,k} = 0 \quad (71)$$

Conservation of fluid momentum

$$\varepsilon_f \rho^f \frac{Dv_k^f}{Dt} = -\varepsilon_f p_{,k}^f + (\lambda^f d_{mn}^f)_{,k} + 2(\mu^f d_{ki}^f)_{,i} + \varepsilon_f \rho^f g_k^f + \pi_{ki}^f \theta_{,i} + R_{ki}^f v_i^f \quad (72)$$

Conservation of solid mass

$$-\frac{\partial \varepsilon_f}{\partial t} + [(1-\varepsilon_f)v_k^f]_{,k} = 0 \quad (73)$$

Conservation of solid momentum

$$(1-\varepsilon_f)\rho_0^s \frac{\partial^2 u_k^s}{\partial t^2} = (\varepsilon_f - 1)p_{,k}^f + [\sigma_{klmn}^s e_{mn}^s]_{,i} + (1-\varepsilon_f)\rho_0^s g_k^s - \pi_{ki}^f \theta_{,i} - R_{ki}^f v_i^f \quad (74)$$

where equation (C2) has been used to replace F_k^s by u_k^s .

Energy equation for the system

$$\begin{aligned} \frac{\varepsilon_f}{\rho^f} \left[(1-\varepsilon_f) \frac{\rho^f \rho_0^s}{\rho} (E^f - E_0^s) + \left(p^f - \theta \frac{\partial p^f}{\partial \theta} \right) \right] \frac{D\rho^f}{Dt} \\ + \rho C_v \frac{D\theta}{Dt} \\ = -p^f (\varepsilon_f v_k^f)_{,k} - p^f v_{k,k}^f + \lambda^f d_{kk}^f d_{ii}^f + 2\mu^f d_{ki}^f d_{ki}^f + \\ \sigma_{klmn}^s e_{kl}^s d_{mn}^s + (\Delta_{klmn}^s d_{mn}^s)_{,k} + [(\kappa^f \delta_{ki} + \kappa_{ki}^s) \theta_{,i}]_{,k} + \\ [(\sigma^f \delta_{ki} + \sigma_{ki}^s) v_i^f]_{,k} - [\varepsilon_f (1-\varepsilon_f) \frac{\rho^f \rho_0^s}{\rho} (E^f - E_0^s) v_k^f]_{,k} - \\ \pi_{ki}^f \theta_{,i} v_k^f - R_{ki}^f v_k^f v_i^f + \rho h \end{aligned} \quad (75)$$

Note that the material coefficients in these equations are macroscopic properties and are different from their microscopic counter parts. For example, the fluid viscosity coefficient μ^f might be zero even if the fluid is microscopically viscous.

Thus far, the balance equations, combined with the constitutive equations, have been reduced to nine equations to be solved for nine unknowns $\rho^f, \varepsilon_f, v_k^f, u_k^s$, and θ . These equations involve the following functions and material coefficients which must be determined experimentally:

$$A^f(\rho^f, \theta), A_0^s(\theta), \sigma_{klmn}^s(\theta), \lambda^f, \mu^f, \kappa^f, \kappa_{ki}^s, \Delta_{klmn}^s, \sigma^f, \sigma_{ki}^s, \pi_{ki}^f, R_{ki}^f \quad (76)$$

Other functions and coefficients such as C_v, E_0^s, E_{klmn}^s are already defined in terms of functions (76). Except for A^f, A_0^s and σ_{klmn}^s which depend on ε_f and θ , these coefficients are functions of ρ^f and ε_f as well as θ . They are subject to the restrictions (C25).

SIMPLIFIED FIELD EQUATIONS — DARCY'S LAW

In the previous section a linear theory was developed which still has retained some degree of generality. It accounts for all inertial, viscous, diffusive, thermal, compressibility and deformability effects. Rarely, however, is one interested in all of these phenomena at the same time. In this section some simplifying assumptions will be made which make the equations more amenable to a practical application of importance.

First, consider the fluid-phase field equations (71) and (72). Assume that the temperature gradient $\theta_{,k}$ will have a significant effect only in heat conduction phenomena, hence

$$\pi_{ki}^f = 0 \quad (77)$$

Then, if the inertial and (macroscopic) viscous effects are neglected, the fluid-phase momentum equation becomes:

$$v_i^f = K_{ik}^f (p_{,k}^f - \rho^f g_k) \quad (78)$$

where it has been assumed that $-R_{ki}^f$ is invertible and according to (C25d), the following restriction holds:

$$\mathbf{K}^f = -\varepsilon_f(\mathbf{R}^f)^{-1} = \mathbf{K}^f(\rho^f, \varepsilon_f, \theta) \geq 0 \quad (79)$$

The condition for $-\mathbf{R}_{kl}^f$ to be invertible is that whenever $v_k^d = 0$ the pore pressure will be hydrostatic and *vice versa*²⁸. This condition is a well known experimental fact.

Thus, equation (78), known as Darcy's law, is shown to be valid as a first approximation for the slow flow of a macroscopically inviscid fluid through a non-uniform anisotropic elastic solid composed of incompressible grains. Equation (78) must be supplemented by the continuity equation (71) as well as the solid phase equations (73) and (74) and the energy equation (75).

Next consider the solid-phase momentum equation (74). If the solid phase is assumed to be isotropic it can be shown²⁵ that t_{kl}^s will be given by an equation, similar in form to the Hooke-Cauchy law of the classical linear theory of an isotropic solid, i.e.

$$t_{kl}^s = -\varepsilon_s p^s \delta_{kl} + \lambda^s e_{mm}^s \delta_{kl} + 2\mu^s e_{kl}^s \quad (80)$$

Substitution of this result into (74), taking into account the assumptions made earlier for the fluid-phase, yields the final form of the solid-phase momentum equation as:

$$(1 - \varepsilon_f) \rho_0^s \frac{\partial^2 u_k^s}{\partial t^2} = (\varepsilon_f - 1) p_{,k}^f + (\lambda^s e_{mm}^s)_{,k} + 2(\mu^s e_{kl}^s)_{,l} + (1 - \varepsilon_f) \rho_0^s g_k + \varepsilon_f (K_{kl}^f)^{-1} v_l^d \quad (81)$$

Finally, the energy equation for the system is considered. The following assumptions and observations are made:

(i) Since the solid is assumed to be isotropic Δ_{kmn}^s must vanish,

$$\Delta_{kmn}^s = 0 \quad (82)$$

(ii) Neglect the work of fluid as well as solid phase stress tensors.

(iii) Neglect the heat conduction due to the motion of fluid with respect to solid, thus

$$\sigma^f \delta_{kl} + \sigma_{kl}^s = 0 \quad (83)$$

(iv) Denote the difference in energy of solid and fluid by ΔE , i.e.

$$\Delta E = \varepsilon_f (1 - \varepsilon_f) \rho_0^f (E^f - E_0^f) / \rho^2 \quad (84)$$

(v) Let κ denote the heat conductivity coefficient for the system; since the solid is isotropic κ_{kl}^s is replaced by $\kappa^s \delta_{kl}$.

$$\kappa = \kappa^f + \kappa^s \quad (85)$$

Hence equation (75) becomes:

$$\frac{1}{\rho^f} \left[\rho \Delta E + \left(p^f - \theta \frac{\partial p^f}{\partial \theta} \right) \right] \frac{D\rho^f}{Dt} + \rho C_v \frac{D\theta}{Dt} = -p^f (\varepsilon_f v_k^d)_{,k} - p^f v_{k,k}^d + (\kappa \theta_{,k} - \rho \Delta E v_{k,k}^d)_{,k} + \varepsilon_f (K_{kl}^f)^{-1} v_k^d v_l^d + \rho h \quad (86)$$

CONCLUSION

Constitutive theory has been applied to develop general equations for flow of a fluid in an elastic solid. The use of ε_f

and $\varepsilon_{f,k}$ as independent parameters has proven to be important in the development of a general theory of flow through porous media. Exploitation of the entropy inequality and imposition of objectivity requirements has led to general equations (64)–(69). Linearization of these equations resulted in equations (71)–(75). Finally the specific assumptions necessary to reduce the linearized fluid-phase momentum balance to Darcy's law are revealed.

APPENDIX A

Exploitation of the entropy inequality

According to the principle of admissibility, the second law shall hold for any set of constitutive equations. Hence, substitution of equations (41) into (34) shall not violate the inequality. This requirement will place some restrictions on the constitutive equations as investigated below.

The following relations will prove useful in the forthcoming manipulations:

$$(i) \quad \frac{D\varepsilon_f}{Dt} = \varepsilon_f d_{kk}^s + v_k^d \varepsilon_{f,k} \quad (A1)$$

which follows from the solid phase mass balance and equations (15), (20b), (24b) and (39)

$$(ii) \quad \frac{D\rho^f}{Dt} = -\frac{\rho^f}{\varepsilon_f} (\varepsilon_s d_{kk}^s + v_k^d \varepsilon_{f,k}) - \rho^f d_{kk}^f \quad (A2)$$

which is obtained from equation (A1) and the fluid phase mass balance:

$$(iii) \quad v_{k,l}^d = d_{kl}^d + \omega_{kl}^d \quad (A3)$$

where ω_{kl}^d is the rotation tensor defined as:

$$(iv) \quad \omega_{kl}^d = -\omega_{lk}^d = v_{[k,l]}^d = \frac{1}{2}(v_{k,l}^d - v_{l,k}^d) \quad (A4)$$

$$(v) \quad v_{k,l}^d = v_{k,l}^f - v_{k,l}^s = d_{kl}^f - d_{kl}^s + \omega_{kl}^f - \omega_{kl}^s \quad (A5)$$

Because A^s in equation (41a) does not depend explicitly on time, the chain rule of differentiation is employed to evaluate $\overset{\circ}{D}A^s/Dt$ which appears in entropy inequality (34):

$$\overset{\circ}{D}A^s = \frac{\partial A^s}{\partial \rho^f} \frac{D\rho^f}{Dt} + \frac{\partial A^s}{\partial \varepsilon_{f,k}} \frac{D\varepsilon_{f,k}}{Dt} + \frac{\partial A^s}{\partial E_{KL}^s} \frac{DE_{KL}^s}{Dt} +$$

$$\frac{\partial A^s}{\partial d_{kl}^f} \frac{Dd_{kl}^f}{Dt} + \frac{\partial A^s}{\partial \theta} \frac{D\theta}{Dt} + \frac{\partial A^s}{\partial \theta_{,k}} \frac{D\theta_{,k}}{Dt} + \frac{\partial A^s}{\partial v_k^d} \frac{Dv_k^d}{Dt} \quad (A6)$$

By use of equations (15), (19a), (20a), (A1)–(A5), and (A6), this equation can be further expanded for each of the phases:

$$\frac{D A^f}{Dt} = \left[-\rho^f \frac{\varepsilon_s}{\varepsilon_f} \frac{\partial A^f}{\partial \rho^f} \delta_{kl} + \frac{\partial A^f}{\partial E_{KL}^f} F_{(kK}^f F_{l)L}^f \right] d_{kl}^f - \rho^f \frac{\partial A^f}{\partial \rho^f} d_{kk}^f -$$

$$\frac{\rho^f \partial A^f}{\varepsilon_f \partial \rho^f} v_k^d \varepsilon_{f,k} + \frac{\partial A^f}{\partial \varepsilon_{f,k}} D \varepsilon_{f,k} + \frac{\partial A^f}{\partial E_{KL}^s} v_k^d E_{KL}^s + \frac{\partial A^f}{\partial d_{kl}^f} \frac{D d_{kl}^f}{Dt} + \frac{\partial A^f}{\partial \theta} \frac{D \theta}{Dt} + \frac{\partial A^f}{\partial \theta_{,k}} \theta_{,k} v_k^f + \frac{\partial A^f}{\partial \theta_{,k}} \frac{D \theta_{,k}}{Dt} + \frac{\partial A^f}{\partial \theta_{,k}} \theta_{,kl} v_k^f + \frac{\partial A^f}{\partial v_k^d} \frac{D v_k^d}{Dt} \quad (A7)$$

$$\frac{D A^s}{Dt} = \left[-\rho^s \frac{\varepsilon_s}{\varepsilon_f} \frac{\partial A^s}{\partial \rho^f} \delta_{kl} + \frac{\partial A^s}{\partial E_{KL}^s} F_{(kK}^s F_{lL}^s + \frac{\partial A^s}{\partial v_{ik}^d} v_i^d \right] d_{kl}^s - \frac{\rho^f \partial A^s}{\varepsilon_f \partial \rho^f} v_k^d \varepsilon_{f,k} - \left[\rho^f \frac{\partial A^s}{\partial \rho^f} \delta_{kl} + \frac{\partial A^s}{\partial v_{ik}^d} v_i^d \right] d_{kl}^f - \frac{\partial A^s}{\partial \rho^f} \rho_{,k}^f v_k^d + \frac{\partial A^s}{\partial d_{kl}^f} \frac{D d_{kl}^f}{Dt} - \frac{\partial A^s}{\partial \varepsilon_{f,k}} \frac{D \varepsilon_{f,k}}{Dt} - \frac{\partial A^s}{\partial \theta_{,k}} \theta_{,kl} + \frac{\partial A^s}{\partial \theta_{,k}} \frac{D \theta_{,k}}{Dt} + \frac{\partial A^s}{\partial \theta_{,k}} \theta_{,kl} v_k^f + \frac{\partial A^s}{\partial v_k^d} \frac{D v_k^d}{Dt} - \frac{\partial A^s}{\partial v_{ik}^d} v_i^d (\omega_{kl}^s - \omega_{kl}^s) \quad (A8)$$

Substitution of equations (A7) and (A8) into equation (34) followed by rearrangement yields:

$$\theta \Gamma = \left[\varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \rho^f} v_k^d \right] \rho_{,k}^f - \left[\varepsilon_f \rho^f \frac{\partial A^f}{\partial E_{KL}^s} v_k^d \right] E_{KL}^s + \left[\varepsilon_s (\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \delta_{kl} - \varepsilon_f \rho^f \frac{\partial A^f}{\partial E_{KL}^s} F_{(kK}^s F_{lL}^s + \frac{\varepsilon_s^2}{\varepsilon_f} \rho^f \rho_0^s \frac{\partial A^s}{\partial \rho^f} \delta_{kl} - \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial E_{KL}^s} F_{(kK}^s F_{lL}^s - \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial v_{ik}^d} v_i^d + t_{kl}^s \right] d_{kl}^s + \left(\varepsilon_f (\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \delta_{kl} + \varepsilon_s \rho_0^s \rho^f \frac{\partial A^s}{\partial \rho^f} \delta_{kl} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial v_{ik}^d} v_i^d + t_{kl}^s \right) d_{kl}^f - \left[\varepsilon_f \rho^f \frac{\partial A^f}{\partial \varepsilon_{f,k}} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \varepsilon_{f,k}} \right] \frac{D \varepsilon_{f,k}}{Dt} + \left[\varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \varepsilon_{f,k}} v_i^d \right] \varepsilon_{f,kl} - \left[\varepsilon_f \rho^f \frac{\partial A^f}{\partial d_{kl}^f} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial d_{kl}^f} \right] \frac{D d_{kl}^f}{Dt} + \left[\varepsilon_s \rho_0^s \frac{\partial A^s}{\partial d_{kl}^f} v_m^d \right] d_{kl,m}^f - \left[\varepsilon_f \rho^f \left(\frac{\partial A^f}{\partial \theta} + S^f \right) + \varepsilon_s \rho_0^s \left(\frac{\partial A^s}{\partial \theta} + S^s \right) \right] \frac{D \theta}{Dt} + \left(\frac{1}{\theta} (q_k^f + q_k^s) - \sum_{\alpha} \varepsilon_{\alpha} \rho^{\alpha} \left(S^{\alpha} + \frac{\partial A^{\alpha}}{\partial \theta} \right) v_k^{\alpha} \right) \theta_{,k} - \left[\varepsilon_f \rho^f \frac{\partial A^f}{\partial \theta_{,k}} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \theta_{,k}} \right] \frac{D \theta_{,k}}{Dt} -$$

$$\left[\varepsilon_f \rho^f \frac{\partial A^f}{\partial \theta_{,k}} v_i^f + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \theta_{,k}} v_i^s \right] \theta_{,kl} - \left[\varepsilon_f \rho^f \frac{\partial A^f}{\partial v_k^d} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial v_k^d} \right] \frac{D v_k^d}{Dt} + \left[\varepsilon_s \rho_0^s \frac{\partial A^s}{\partial v_{ik}^d} v_i^d \right] (\omega_{kl}^f - \omega_{kl}^s) + \left((\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \varepsilon_{f,k} + \frac{\varepsilon_s}{\varepsilon_f} \rho^f \rho_0^s \frac{\partial A^s}{\partial \rho^f} \right) \varepsilon_{f,k} - \varepsilon_f \rho^f \hat{T}_k^f \Big) v_k^d \geq 0 \quad (A9)$$

Now, according to the constitutive equations (41), none of the constitutive functions depend on $\rho_{,k}^f$, E_{KL}^s , d_{kl}^s , $D \varepsilon_{f,k}/Dt$, $\varepsilon_{f,kl}$, $D \theta/Dt$, $D^f d_{kl}^f/Dt$, $\theta_{,kl}$, $D v_k^d/Dt$, ω_{kl}^f and ω_{kl}^s . Therefore equation (A9) indicates that $\theta \Gamma$ is a linear function of these quantities. The necessary and sufficient condition for Γ to be non-negative for all independent thermodynamic states is that the coefficients of the above-mentioned quantities must vanish. Hence all the bracketed terms in equation (A9) must be equal to zero. Therefore

$$\frac{\partial A^s}{\partial \rho^f} = 0 \quad (A10)$$

$$\frac{\partial A^f}{\partial E_{KL}^s} = 0 \quad (A11)$$

and

$$t_{kl}^s = -\varepsilon_s (\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \delta_{kl} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial E_{KL}^s} F_{(kK}^s F_{lL}^s + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial v_{ik}^d} v_i^d \quad (A12)$$

where equation (A11) has been employed to simplify the form of equation (A12). Furthermore,

$$\varepsilon_f \rho^f \frac{\partial A^f}{\partial \varepsilon_{f,k}} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \varepsilon_{f,k}} = 0 \quad (A13a)$$

and

$$\frac{\partial A^s}{\partial \varepsilon_{f,k}} v_i^d = 0 \quad (A13b)$$

where the fact that $\varepsilon_{f,kl}$ is symmetric has been used. Additionally:

$$\varepsilon_f \rho^f \frac{\partial A^f}{\partial d_{kl}^f} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial d_{kl}^f} = 0 \quad (A14a)$$

and

$$\frac{\partial A^s}{\partial d_{kl}^f} v_m^d = 0 \quad (A14b)$$

which, when combined, indicate that:

$$\frac{\partial A^\alpha}{\partial d_{kl}^f} = 0 \quad \alpha = f, s \quad (\text{A14c})$$

Also,

$$\varepsilon_f \rho^f S^f + \varepsilon_s \rho_0^s S^s = -\varepsilon_f \rho^f \frac{\partial A^f}{\partial \theta} - \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \theta} \quad (\text{A15})$$

$$\varepsilon_f \rho^f \frac{\partial A^f}{\partial \theta_{,k}} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \theta_{,k}} = 0 \quad (\text{A16a})$$

and

$$\varepsilon_f \rho^f \frac{\partial A^f}{\partial \theta_{,ik}} v_i^f + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial \theta_{,ik}} v_i^s = 0 \quad (\text{A16b})$$

where the fact that $\theta_{,kl}$ is symmetric has been accounted for. Finally

$$\varepsilon_f \rho^f \frac{\partial A^f}{\partial v_k^d} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial v_k^d} = 0 \quad (\text{A17})$$

and

$$\left(\frac{\partial A^s}{\partial v_k^d} v_i^d - \frac{\partial A^s}{\partial v_i^d} v_k^d \right) = 0 \quad (\text{A18})$$

The residual inequality, composed of the surviving terms in equation (A9) will be:

$$\begin{aligned} \theta \Gamma = & \left(\varepsilon_f (\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \delta_{kl} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial v_{ik}^d} v_i^d + t_{kl}^f \right) d_{kl}^f + \\ & \left(\frac{1}{\theta} (q_k^f + q_k^s) - \varepsilon_f \rho^f \left(S^f + \frac{\partial A^f}{\partial \theta} \right) v_k^d \right) \theta_{,k} + \left((\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \varepsilon_{f,k} - \right. \\ & \left. \varepsilon_f \rho^f \hat{T}_k^f \right) v_k^d \geq 0 \quad (\text{A19}) \end{aligned}$$

Equations (A10)–(A18) may be further investigated to extract more information. Consider equation (A18) which may be rearranged as:

$$\frac{\partial(A^s v_i^d)}{\partial v_k^d} - \frac{\partial(A^s v_k^d)}{\partial v_i^d} = 0 \quad (\text{A20})$$

A particular solution for this equation is:

$$A^s v_i^d = \Lambda_{ki}^s v_k^d + \lambda_i^s \quad (\text{A21a})$$

with

$$\Lambda_{ki}^s = \Lambda_{ik}^s \quad (\text{A21b})$$

where Λ_{ki}^s and λ_i^s are not functions of v_k^d . Because the right hand side is linear in v_k^d , the left hand side must also be linear. Therefore, A^s must not depend on v_k^d . From equation (A17) one can also show that A^f does not depend on v_k^d so that:

$$\frac{\partial A^f}{\partial v_k^d} = \frac{\partial A^s}{\partial v_k^d} = 0 \quad (\text{A22})$$

From this last equation and the fact that v_k^d and $\varepsilon_{f,k}$ are independent variables, equations (A13) yield:

$$\frac{\partial A^s}{\partial \varepsilon_{f,k}} = \frac{\partial A^f}{\partial \varepsilon_{f,k}} = 0 \quad (\text{A23})$$

Similarly using equation (A22) in conjunction with equation (A16), one finds that:

$$\frac{\partial A^s}{\partial \theta_{,k}} = \frac{\partial A^f}{\partial \theta_{,k}} = 0 \quad (\text{A24})$$

Equations (A10), (A11), (A14c), (A22), (A23), and (A24) demonstrate that A^f and A^s do not depend on some of the independent variables. Hence equation (41a) is simplified to the form:

$$A^f = A^f(\rho^f, \theta) \quad (\text{A25a})$$

and

$$A^s = A^s(E_{kb}^s, \theta) \quad (\text{A25b})$$

because of the dependence of A on A^f and A^s given by equation (36), it is also clear that:

$$A = A(\rho^f, E_{kb}^s, \theta) \quad (\text{A25c})$$

From the definitions of A and S given in equation (36), it can be seen that equation (A15) may be written as:

$$S = -\frac{\partial A}{\partial \theta} \quad (\text{A26a})$$

Then from equation (A25c):

$$S = S(\rho^f, E_{kb}^s, \theta) \quad (\text{A26b})$$

The internal energy function E has been defined in equation (35) to be equal to $A + \theta S$. Thus from equations (A25c) and (A26b):

$$E = E(\rho^f, E_{kb}^s, \theta) \quad (\text{A27})$$

Because A^s does not depend on v_k^d , equation (A12) reduces to:

$$t_{kl}^s = -\varepsilon_s (\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \delta_{kl} + \varepsilon_s \rho_0^s \frac{\partial A^s}{\partial E_{KL}^s} F_{(kk}^s F_{l)L}^s \quad (\text{A28})$$

These results, (A25)–(A28), are useful simplifications of constitutive equations (41) obtained from the second law of thermodynamics which has been reduced to the form given in equation (A19).

APPENDIX B

Expansion of DE/Dt

Consider the following relations obtained and/or defined previously:

$$A^f = A^f(\rho^f, \theta) \quad (\text{B1})$$

$$A^s = A^s(E_{kb}^s, \theta) \quad (\text{B2})$$

$$A = \frac{1}{\rho} [\varepsilon_f \rho^f A^f + \varepsilon_s \rho_0^s A^s] = A(\rho^f, E_{kl}^s, \theta) \quad (\text{B3})$$

$$S = -\frac{\partial A}{\partial \theta} = S(\rho^f, E_{kl}^s, \theta) \quad (\text{B4})$$

$$E = A + \theta S = A - \theta \frac{\partial A}{\partial \theta} = E(\rho^f, E_{kl}^s, \theta) \quad (\text{B5})$$

Also on the basis of equations (29), (30) and (41a, b) one may write:

$$E^\alpha = E^\alpha(\rho^f, \varepsilon_{f,k}, E_{kl}^s, d_{kl}^f, \theta, \theta_{,k}, v_k^f) \quad \alpha = f, s \quad (\text{B6})$$

and

$$E = \frac{1}{\rho} \sum_{\alpha=f,s} \varepsilon_\alpha \rho^\alpha E^\alpha \quad (\text{B7})$$

From equation (B5) and the chain rule of differentiation, it follows that:

$$dE = \frac{\partial E}{\partial \rho^f} d\rho^f + \frac{\partial E}{\partial E_{kl}^s} dE_{kl}^s + \frac{\partial E}{\partial \theta} d\theta \quad (\text{B8})$$

The task here is to find

$$\frac{\partial E}{\partial \rho^f}, \quad \frac{\partial E}{\partial E_{kl}^s}, \quad \frac{\partial E}{\partial \theta}.$$

However, in thermodynamics, $\partial E/\partial \theta$ is known as the heat capacity and is denoted by C_v . According to equation (B7), this can also be written as:

$$C_v = C_v(\rho^f, E_{kl}^s, \theta) = \frac{\partial E}{\partial \theta} = \sum_{\alpha=f,s} \varepsilon_\alpha \rho^\alpha \frac{\partial E^\alpha}{\partial \theta} \quad (\text{B9})$$

hence

$$C_v = \frac{1}{\rho} \sum_{\alpha=f,s} \varepsilon_\alpha \rho^\alpha C_v^\alpha \quad (\text{B10})$$

where C_v^α ($\alpha = f, s$) is the heat capacity for the α th phase. So, only $\partial E/\partial \rho^f$ and $\partial E/\partial E_{kl}^s$ need to be determined. From equation (B5), since ρ^f and E_{kl}^s are independent of θ ,

$$\begin{aligned} \frac{\partial E}{\partial \rho^f} &= \frac{\partial A}{\partial \rho^f} - \theta \frac{\partial}{\partial \theta} \left(\frac{\partial A}{\partial \rho^f} \right) \\ \frac{\partial E}{\partial E_{kl}^s} &= \frac{\partial A}{\partial E_{kl}^s} - \theta \frac{\partial}{\partial \theta} \left(\frac{\partial A}{\partial E_{kl}^s} \right) \end{aligned} \quad (\text{B11})$$

Now, $\partial A/\partial \rho^f$ and $\partial A/\partial E_{kl}^s$ are needed to determine $\partial E/\partial \rho^f$ and $\partial E/\partial E_{kl}^s$. From equation (B3),

$$\begin{aligned} dA &= \frac{1}{\rho} [d(\rho A) - A d\rho] = \frac{1}{\rho} [d(\varepsilon_f \rho^f A^f + \varepsilon_s \rho_0^s A^s) \\ &\quad - A d(\varepsilon_f \rho^f + \varepsilon_s \rho_0^s)] \end{aligned} \quad (\text{B12})$$

which, after rearrangement, becomes:

$$\begin{aligned} \rho dA &= [\rho^f (A^f - A) + \rho_0^s (A^s - A^s)] d\varepsilon_f + \varepsilon_f (A^f - A) d\rho^f + \\ &\quad \varepsilon_f \rho^f dA^f + \varepsilon_s \rho_0^s dA^s \end{aligned} \quad (\text{B13})$$

From equations (B1), (B2), and (54) it is found that:

$$dA^f = \frac{p^f}{(\rho^f)^2} d\rho^f + \frac{\partial A^f}{\partial \theta} d\theta \quad (\text{B14a})$$

$$dA^s = \frac{\partial A^s}{\partial E_{kl}^s} dE_{kl}^s + \frac{\partial A^s}{\partial \theta} d\theta \quad (\text{B14b})$$

Then, from equations (8) and (40b), and after some manipulation, it can be proven that:

$$d\varepsilon_f = \varepsilon_s B_{kl}^s dE_{kl}^s \quad (\text{B15a})$$

where B_{kl}^s is the Piola-Kirchhoff deformation tensor defined as^{2,5}

$$B_{kl}^s = X_{k,\lambda}^s X_{l,\lambda}^s \quad (\text{B15b})$$

Also from equation (B3) it is found that:

$$A^f - A = \frac{\varepsilon_s \rho_0^s}{\rho} (A^f - A^s) \quad (\text{B16a})$$

$$A - A^s = \frac{\varepsilon_f \rho^f}{\rho} (A^f - A^s) \quad (\text{B16b})$$

Substitution of equations (B14)-(B16) into equation (B13), after rearrangement, yields:

$$\begin{aligned} dA &= \frac{\varepsilon_s \rho_0^s}{\rho} \left[\frac{\rho^f}{\rho} (A^f - A^s) B_{kl}^s + \frac{\partial A^s}{\partial E_{kl}^s} \right] dE_{kl}^s + \\ &\quad \frac{\varepsilon_f}{\rho} \left[\frac{\varepsilon_s \rho_0^s}{\rho} (A^f - A^s) + p^f/\rho^f \right] d\rho^f - S d\theta \end{aligned} \quad (\text{B17})$$

where the relationship between S and $\partial A^s/\partial \theta$, equations (B3) and (B4), has been exploited. Now from equations (B11) and (B17), it readily follows that:

$$\begin{aligned} \frac{\partial E}{\partial \rho^f} &= \frac{\varepsilon_f}{\rho} \left[\frac{\varepsilon_s \rho_0^s}{\rho} \left(\left(A^f - \theta \frac{\partial A^f}{\partial \theta} \right) - \left(A^s - \theta \frac{\partial A^s}{\partial \theta} \right) \right) + \right. \\ &\quad \left. \frac{1}{\rho^f} \left(p^f - \theta \frac{\partial p^f}{\partial \theta} \right) \right] \end{aligned} \quad (\text{B18a})$$

$$\begin{aligned} \frac{\partial E}{\partial E_{kl}^s} &= \frac{\varepsilon_s \rho_0^s}{\rho} \left[\frac{\rho^f}{\rho} \left(\left(A^f - \theta \frac{\partial A^f}{\partial \theta} \right) - \left(A^s - \theta \frac{\partial A^s}{\partial \theta} \right) \right) B_{kl}^s + \right. \\ &\quad \left. \frac{\partial A^s}{\partial E_{kl}^s} - \theta \frac{\partial}{\partial \theta} \left(\frac{\partial A^s}{\partial E_{kl}^s} \right) \right] \end{aligned} \quad (\text{B18b})$$

On the other hand, from equations (B6) and (B7), and the relation $\partial \varepsilon_f/\partial E_{kl}^s = \varepsilon_s B_{kl}^s$ obtained from equation (B15a), it can be proven that:

$$\frac{\partial E}{\partial E_{kl}^s} = \frac{\varepsilon_s \rho_0^s}{\rho} \left[\frac{\rho^f}{\rho} (E^f - E^s) B_{kl}^s + \frac{\varepsilon_f \rho^f}{\varepsilon_s \rho_0^s} \frac{\partial E^f}{\partial E_{kl}^s} + \frac{\partial E^s}{\partial E_{kl}^s} \right] \quad (\text{B19})$$

Additionally, from equations (B3), (B5), and (B7) one finds that:

$$\varepsilon_f \rho^f E^f + \varepsilon_s \rho_0^s E^s = \varepsilon_f \rho^f \left(A^f - \theta \frac{\partial A^f}{\partial \theta} \right) + \varepsilon_s \rho_0^s \left(A^s - \theta \frac{\partial A^s}{\partial \theta} \right) \quad (\text{B20})$$

Combination of equations (B18b), (B19), and (B20) results in:

$$E^s B_{KL}^s - \frac{\varepsilon_f \rho^f}{\varepsilon_s \rho_0^s} \frac{\partial E^f}{\partial E_{KL}^s} - \varepsilon_f \frac{\partial E^s}{\partial E_{KL}^s} = \left(A^s - \theta \frac{\partial A^s}{\partial \theta} \right) B_{KL}^s - \varepsilon_f \frac{\partial}{\partial E_{KL}^s} \left(A^s - \theta \frac{\partial A^s}{\partial \theta} \right) \quad (\text{B21})$$

A particular solution to this equation is:

$$\frac{\partial E^f}{\partial E_{KL}^s} = 0 \quad (\text{B22})$$

and

$$E^s = A^s - \theta \frac{\partial A^s}{\partial \theta} \quad (\text{B23a})$$

which upon substitution into equation (B20) yields:

$$E^f = A^f - \theta \frac{\partial A^f}{\partial \theta} \quad (\text{B23b})$$

Combination of equations (B23) and (29) (with $\theta^\alpha = \theta$) yields:

$$S^\alpha = - \frac{\partial A^\alpha}{\partial \theta} \quad \alpha = f, s \quad (\text{B24})$$

Hence, equations (B18) may be put in the following form:

$$\frac{\partial E}{\partial \rho^f} = \frac{\varepsilon_f}{\rho} \left[\frac{\varepsilon_s \rho_0^s}{\rho} (E^f - E^s) + \frac{1}{\rho^f} \left(p^f - \theta \frac{\partial p^f}{\partial \theta} \right) \right] \quad (\text{B25a})$$

$$\frac{\partial E}{\partial E_{KL}^s} = \frac{\varepsilon_s \rho_0^s}{\rho} \left[\frac{\rho^f}{\rho} (E^f - E^s) B_{KL}^s + \frac{\partial E^s}{\partial E_{KL}^s} \right] \quad (\text{B25b})$$

Finally substitution of equation (B25) into (B8) yields:

$$\rho \frac{DE}{Dt} = \varepsilon_f \left[\frac{\varepsilon_s \rho_0^s}{\rho} (E^f - E^s) + \frac{1}{\rho^f} \left(p^f - \theta \frac{\partial p^f}{\partial \theta} \right) \right] \frac{D\rho^f}{Dt} + \varepsilon_s \rho_0^s \left[\frac{\rho^f}{\rho} (E^f - E^s) B_{KL}^s + \frac{\partial E^s}{\partial E_{KL}^s} \right] \frac{DE_{KL}^s}{Dt} + \rho C_v \frac{D\theta}{Dt} \quad (\text{B26})$$

The physical significance of different terms in equation (B26) is discussed in the section on constitutive equations.

Note that relations (B22) and (B23) are not, by any means essential to the developments of this paper and have been adopted only for the sake of simplicity.

APPENDIX C

Linearization of the constitutive equations

As explained earlier, the constitutive equations will be expanded in a linear Taylor series around $Z_A = 0$ where:

$$\{Z_A, A = 1, 21\} = \{\varepsilon_{f,k}, e_{kl}^s, d_{kl}^f, \theta, v_k^d\}$$

The solid-phase infinitesimal strain tensor, e_{kl}^s , is given below in terms of the solid-phase displacement vector, u_k^s ,

$$e_{kl}^s = \frac{1}{2}(u_{k,l}^s + u_{l,k}^s) \quad (\text{C1})$$

$$u_k^s = x_k - X_k^s \delta_{Kk} \quad (\text{C2})$$

where δ_{Kk} is the shifter (for its definition and properties see Eringen²⁵).

Consider the solid-phase free energy A^s and stress tensor t_{kl}^s given by equations (55b) and (58b), respectively. The procedure of obtaining a linear expression for t_{kl}^s is exactly analogous to that given in Eringen²⁵ for a single-phase solid continuum and will not be repeated here. The results are:

$$A^s = A_0^s(\varepsilon_f, \theta) + \frac{1}{2\varepsilon_s \rho_0^s} \sigma_{klmn}^s(\varepsilon_f, \theta) e_{kl}^s e_{mn}^s \quad (\text{C3})$$

$$t_{kl}^s = -\varepsilon_s p^f(\rho^f, \theta) \delta_{kl} + \sigma_{klmn}^s(\varepsilon_f, \theta) e_{mn}^s \quad (\text{C4})$$

where

$$\sigma_{klmn}^s = 2\varepsilon_s \rho_0^s \left(\frac{\partial^2 A^s}{\partial e_{kl}^s \partial e_{mn}^s} \right)_0 \quad (\text{C5})$$

is equivalent to the elasticity coefficient in Hooke's law. The subscript 0 in equation (C5) and hereafter indicates that the parenthesized function is being evaluated at $Z_A = 0$, $A = 1, 21$. In arriving at equation (C4), it has been assumed that the natural state of the solid phase, in the absence of fluid-phase pore pressure, is stress free.

Note that linearization in terms of E_{KL}^s removes the implicit dependence of the constitutive functions on ε_f . This dependence must be resumed by replacing ε_f back in the list of independent variables. Now consider the non-equilibrium part of the fluid-phase stress tensor τ_{kl}^f . It can be approximated by the following expression:

$$\tau_{kl}^f = (\tau_{kl}^f)_0 + \left(\frac{\partial \tau_{kl}^f}{\partial e_{mn}^s} \right)_0 e_{mn}^s + \left(\frac{\partial \tau_{kl}^f}{\partial \varepsilon_{f,m}} \right)_0 \varepsilon_{f,m} +$$

$$\left(\frac{\partial \tau_{kl}^f}{\partial d_{mn}^f} \right)_0 d_{mn}^f + \left(\frac{\partial \tau_{kl}^f}{\partial \theta} \right)_0 \theta + \left(\frac{\partial \tau_{kl}^f}{\partial v_m^d} \right)_0 v_m^d \quad (\text{C6})$$

This expression is subject to the equilibrium restriction (61), i.e.

$$\tau_{kl;l}^f = (\tau_{kl}^f)_0 + \left(\frac{\partial \tau_{kl}^f}{\partial e_{mn}^s} \right)_0 e_{mn}^s + \left(\frac{\partial \tau_{kl}^f}{\partial \varepsilon_{f,m}} \right)_0 \varepsilon_{f,m} = 0 \quad (\text{C7})$$

which must hold for all values of e_{mn}^s and $\varepsilon_{f,m}$. It follows that:

$$(\tau_{kl}^f)_0 = \left(\frac{\partial \tau_{kl}^f}{\partial e_{mn}^s} \right)_0 = \left(\frac{\partial \tau_{kl}^f}{\partial \varepsilon_{f,m}} \right)_0 = 0 \quad (\text{C8})$$

Therefore equation (C6) becomes:

$$\tau_{kl}^f = \sigma_{klmn}^f d_{mn}^f + \kappa_{klm}^f \theta + K_{klm}^f v_m^d \quad (\text{C9})$$

where

$$\sigma_{klmn}^f(\rho^f, \varepsilon_f, \theta) = \left(\frac{\partial \tau_{kl}^f}{\partial d_{mn}^f} \right)_0, \quad \kappa_{klm}^f(\rho^f, \varepsilon_f, \theta) = \left(\frac{\partial \tau_{kl}^f}{\partial \theta_{,m}} \right)_0$$

and

$$K_{klm}^f(\rho^f, \varepsilon_f, \theta) = \left(\frac{\partial \tau_{kl}^f}{\partial v_m^d} \right)_0$$

are material coefficients.

The objectivity requirements, when applied to the constitutive equations for a fluid, will be met only if the constitutive functions are isotropic. In other words, if there exists no preferred direction in the fluid, it shall be considered as an isotropic material. For isotropic materials the material coefficients must be isotropic. Therefore σ_{klmn}^f , κ_{klm}^f , K_{klm}^f must be isotropic tensors whose most general forms can be proven to be²⁹:

$$\sigma_{klmn}^f = \lambda^f \delta_{kl} \delta_{mn} + \mu^f (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}) \quad (C10)$$

$$\kappa_{klm}^f = K_{klm}^f = 0 \quad (C11)$$

Equation (C9) reduces to the familiar form:

$$\tau_{ki}^f = \lambda^f d_{mm}^f \delta_{ki} + 2\mu^f d_{ki}^f \quad (C12)$$

where λ^f and μ^f , called viscosity coefficients, are functions of ρ^f , ε_f and θ .

Next consider the non-equilibrium part of momentum exchange term $\hat{\tau}_k^f$.

$$\begin{aligned} \hat{\tau}_k^f = & (\hat{\tau}_k^f)_0 + \left(\frac{\partial \hat{\tau}_k^f}{\partial e_{mn}^s} \right)_0 e_{mn}^s + \left(\frac{\partial \hat{\tau}_k^f}{\partial d_{mn}^f} \right)_0 d_{mn}^f + \left(\frac{\partial \hat{\tau}_k^f}{\partial \varepsilon_{f,i}} \right)_0 \varepsilon_{f,i} + \\ & \left(\frac{\partial \hat{\tau}_k^f}{\partial \theta_{,i}} \right)_0 \theta_{,i} + \left(\frac{\partial \hat{\tau}_k^f}{\partial v_i^d} \right)_0 v_i^d \end{aligned} \quad (C13)$$

This expression must be subject to the equilibrium restriction (64). In a similar manner as in the case of τ_{ki}^f , the following result will be obtained:

$$\hat{\tau}_k^f = \pi_{ki}^f \theta_{,i} + R_{ki}^f v_k^d \quad (C14)$$

where

$$\pi_{ki}^f = \left(\frac{\partial \hat{\tau}_k^f}{\partial \theta_{,i}} \right)_0 \quad \text{and} \quad R_{ki}^f = \left(\frac{\partial \hat{\tau}_k^f}{\partial v_i^d} \right)_0$$

are functions of ρ^f , ε_f , and θ .

The third order tensor $(\partial \hat{\tau}_k^f / \partial d_{mn}^f)_0$ has been set equal to zero on the grounds that there exists no isotropic third order tensor.

Next consider the solid-phase internal energy function, E^s . Combination of equations (57b) and (C3) yields:

$$E^s = E_0^s + \frac{1}{2\varepsilon_s \rho_0^s} E_{klmn}^s \quad (\text{or read } E_{klmn}^s(\varepsilon_f, \theta)) \quad (C15)$$

where

$$E_0^s(\varepsilon_f, \theta) = A_0^s - \theta \frac{\partial A_0^s}{\partial \theta} \quad (C16a)$$

$$E_{klmn}^s(\varepsilon_f, \theta) = \sigma_{klmn}^s - \theta \frac{\partial \sigma_{klmn}^s}{\partial \theta} \quad (C16b)$$

Substitution of equation (C15) into (B26) and neglect of all second order terms yields:

$$\begin{aligned} \frac{DE}{Dt} = & \frac{\varepsilon_f}{\rho \rho^f} \left[\varepsilon_s \frac{\rho^f \rho_0^s}{\rho} (E^s - E_0^s) + \left(p^f - \theta \frac{\partial p^f}{\partial \theta} \right) \right] \frac{D\rho^f}{Dt} + \\ & C_v(\rho^f, \varepsilon_f, \theta, e_{ki}^s) \frac{D\theta}{Dt} \end{aligned} \quad (C17)$$

The coefficient C_v has been left as a function of e_{ki}^s and has not been linearized at this point.

Now consider the Taylor's expansions for q_k^f and q_k^s .

$$\begin{aligned} q_k^s = & (q_k^s)_0 + \left(\frac{\partial q_k^s}{\partial \varepsilon_{f,i}} \right)_0 \varepsilon_{f,i} + \left(\frac{\partial q_k^s}{\partial e_{mn}^s} \right)_0 e_{mn}^s + \left(\frac{\partial q_k^s}{\partial d_{mn}^f} \right)_0 d_{mn}^f + \\ & \left(\frac{\partial q_k^s}{\partial \theta_{,i}} \right)_0 \theta_{,i} + \left(\frac{\partial q_k^s}{\partial v_i^d} \right)_0 v_i^d \quad \alpha = f, s \end{aligned} \quad (C18)$$

These equations must satisfy the equilibrium condition equation (62). By similar arguments as in the case of τ_{ki}^f it will be concluded that:

$$(q_k^f)_0 + (q_k^s)_0 = 0 \quad (C19)$$

$$\left(\frac{\partial q_k^f}{\partial \varepsilon_{f,i}} \right)_0 + \left(\frac{\partial q_k^s}{\partial \varepsilon_{f,i}} \right)_0 = 0 \quad (C19b)$$

$$\left(\frac{\partial q_k^f}{\partial e_{mn}^s} \right)_0 + \left(\frac{\partial q_k^s}{\partial e_{mn}^s} \right)_0 = 0 \quad (C19c)$$

Furthermore, due to isotropy requirements for the fluid phase, the following restrictions will apply:

$$(q_k^f)_0 = 0 \quad (C20a)$$

$$\left(\frac{\partial q_k^f}{\partial d_{mn}^f} \right)_0 = 0 \quad (C20b)$$

$$\left(\frac{\partial q_k^f}{\partial \theta_{,i}} \right)_0 = \kappa^f \delta_{ki} = \kappa^f(\rho^f, \varepsilon_f, \theta) \delta_{ki} \quad (C20c)$$

$$\left(\frac{\partial q_k^f}{\partial v_i^d} \right)_0 = \sigma^f \delta_{ki} = \sigma^f(\rho^f, \varepsilon_f, \theta) \delta_{ki} \quad (C20d)$$

Also the following notation is employed:

$$\Delta_{kmn}^s = \left(\frac{\partial q_k^s}{\partial d_{mn}^f} \right)_0 = \Delta_{kmn}^s(\rho^f, \varepsilon_f, \theta) \quad (C21a)$$

$$\kappa_{ki}^s = \left(\frac{\partial q_k^s}{\partial \theta_{,i}} \right)_0 = \kappa_{ki}^s(\rho^f, \varepsilon_f, \theta) \quad (C21b)$$

$$\sigma_{ki}^s = \left(\frac{\partial q_k^s}{\partial v_i^d} \right)_0 = \sigma_{ki}^s(\rho^f, \varepsilon_f, \theta) \quad (C21c)$$

Substitution of equations (C19)–(C21) back into equation (C18) results in:

$$q_k^f + q_k^s = \Delta_{kmn}^s d_{mn}^f + (\kappa^f \delta_{ki} + \kappa_{ki}^s) \theta_{,i} + (\sigma^f \delta_{ki} + \sigma_{ki}^s) v_i^d \quad (C22)$$

The linearized equations developed in this Appendix must not violate the second law of thermodynamics. Hence, substitution of equations (C12), (C14) and (C22) into (64) and rearrangement yields:

$$\theta\Gamma = (\lambda^f + \frac{2}{3}\mu^f)d_{kk}^f d_{ll}^f + 2\mu^f d_{ki}^f d_{kl}^f + \frac{1}{\theta}\Delta_{kmn}^f d_{mn}^f \theta_{,k} + \frac{1}{\theta}(\kappa^f \delta_{kl} + \kappa_{kl}^s)\theta_{,k}\theta_{,l} + \left[\frac{1}{\theta}(\sigma^f \delta_{kl} + \sigma_{kl}^s) - \pi_{kl}^f \right] v_k^d \theta_{,l} - R_{kl}^f v_k^d v_l^d \geq 0 \quad (C23)$$

where d_{kl}^f is the deviatoric part of d_{kl}^f , i.e.

$$d_{kl}^f = d_{kl}^f - \frac{1}{3}d_{mn}^f \delta_{kl} \quad (C24)$$

The inequality (C23) shall hold for all values of $\varepsilon_{f,k}$, d_{kl}^f , $\theta_{,k}$, and v_k^d . Because none of the material coefficients depend on these quantities, the following results will be obtained:

$$\lambda^f + \frac{2}{3}\mu^f \geq 0 \quad (C25a)$$

$$\mu^f \geq 0 \quad (C25b)$$

$$\kappa^f \delta_{kl} + \kappa_{kl}^s \geq 0 \quad (C25c)$$

$$-R_{kl}^f \geq 0 \quad (C25d)$$

Additionally the matrix of coefficients of the terms involving d_{kk}^f , d_{kl}^f , $\theta_{,k}$, and v_k^d must be positive definite. This condition will yield a complicated expression and will not be explored here. The results of this Appendix may be summarized as follows:

$$A^f = A^f(\rho^f, \theta) \quad (C26a)$$

$$E^f = E^f(\rho^f, \theta) \quad (C26b)$$

$$A^s = A_0^s(\varepsilon_f, \theta) + \frac{1}{2\rho_0^s} \sigma_{klmn}^s(\varepsilon_f, \theta) e_{kl}^s e_{mn}^s \quad (C27a)$$

$$E^s = E_0^s + \frac{1}{2\rho_0^s} E_{klmn}^s e_{kl}^s e_{mn}^s \quad (C27b)$$

where E_0^s and E_{klmn}^s are given by equations (C16a, b), respectively.

$$\tau_{kl}^f = -p^f(\rho^f, \theta)\delta_{kl} + \sigma_{klmn}^f(\varepsilon_f, \theta)_n e_{mn}^f \quad (C28)$$

$$\tau_{kl}^f = \lambda^f d_{mn}^f \delta_{kl} + 2\mu^f d_{kl}^f \quad (C29)$$

$$\hat{\tau}_k^f = \pi_{kl}^f \theta_{,l} + R_{kl}^f v_l^d \quad (C30)$$

$$q_k^f + q_k^s = \Delta_{kmn}^s d_{mn}^f + (\kappa^f \delta_{kl} + \kappa_{kl}^s)\theta_{,l} + (\sigma^f \delta_{kl} + \sigma_{kl}^s)v_l^d \quad (C31)$$

where the material coefficients are all functions of ρ^f , ε_f , and θ , unless otherwise indicated. They are also subject to restrictions (C25).

NOMENCLATURE

| | |
|--------------------|---|
| A | free energy density function for the porous medium, |
| A^α | α -phase free energy density function, |
| A_0^s | residual free energy of the solid phase; see equation (C3), |
| B_{KL}^s | Piola-Kirchhoff deformation tensor defined in equation (B15b), |
| C_v | heat capacity of the system, |
| C_v^α | α -phase heat capacity, |
| d_{kl}^s | α -phase strain rate tensor; defined in equation (20b), |
| d_{kl}^f | deviatoric part of d_{kl}^f , defined in equation (C24), |
| e_{kl}^s | infinitesimal strain tensor; defined in equation (C1), |
| E | internal energy density function of the system; see equation (32), |
| E_0^s | residual energy of the solid phase; see equation (C16a), |
| E^α | α -phase internal energy density function, |
| E_{KL}^s | solid-phase strain tensor; see equation (10), |
| E_{klmn}^s | material coefficient; see equation (C16b), |
| f, f | tensorial quality, |
| F^s, F_k^s | a function defining the motion of solid phase, deformation gradient of solid phase: see equation (9), |
| F_{kk}^s | deformation gradient of solid phase: see equation (9), |
| g | gravity vector, |
| g_k^α | α -phase external supply of momentum, |
| h | external supply of energy to the system; defined in equations (33), |
| h^α | α -phase external supply of energy, |
| J^s | Jacobian of the solid-phase motion; defined in equation (8), |
| K, K^f | permeability tensor, |
| K_{klm}^f | material coefficient; see equation (C9), |
| p^f | fluid-phase thermodynamic pressure, |
| q^α | α -phase heat flux vector, |
| Q^α | exchange of energy due to mechanical interactions, |
| R^f, R_{kl}^f | resistivity tensor, |
| S | internal entropy density function of the system; equation (36b), |
| S^α | α -phase internal entropy density function, |
| t | time, |
| t_{kl}^α | α -phase stress tensor, |
| \hat{T}_s^α | exchange of momentum due to mechanical interactions, |
| u_k^s | solid-phase displacement vector; defined in equation (C2), |
| $v \cdot$ | velocity field of the system, |
| v^d | relative velocity of the fluid with respect to the solid, |
| x, x_k | position vector in the spatial frame, |
| X^s, X_k^s | position vector in the solid-phase material frame, and |
| Z_A | elements of a set of variables. |
| <i>Greek</i> | |
| Γ | net rate of production of entropy of the system, |
| Γ^α | net rate of production of entropy of α th phase, |
| δ_{kl} | Kronecker delta in the spatial frame, |
| δ_{KL} | Kronecker delta in the material frame, |
| δ_{KL} | shifter, |
| ΔE | defined in equation (84), |
| Δ_{kmn}^s | material coefficient in equation (C21a), |

| | |
|----------------------|--|
| ε_α | α -phase volume fraction, |
| θ | temperature, |
| κ | heat conductivity of the system, |
| κ^f | material coefficient in equation (C20c), |
| κ^s | material coefficient in equation (85), |
| κ_{kl}^s | material coefficient in equation (C21b), and |
| κ_{klm}^f | material coefficient in equation (C9), |
| λ^f | material coefficient in equation (C10), |
| λ^s | material coefficient in equation (80), |
| μ^f | material coefficient in equation (C10), |
| μ^s | material coefficient in equation (80), |
| v^α | velocity of α -phase relative to the velocity of the system; defined in equation (19b), |
| π_{kl}^f | material coefficient in equation (C14), |
| ρ | mass density of the system, |
| ρ^α | α -phase intrinsic mass density, |
| ρ_α | α -phase bulk mass density, |
| ρ_0^s | initial mass density of the solid phase, |
| σ^f | material coefficient in equation (C20d), |
| σ_{kl}^s | material coefficient in equation (C21c), |
| σ_{klmn}^f | material coefficient in equation (C9), |
| σ_{klmn}^s | material coefficient in equation (C3), |
| $\hat{\tau}_k^f$ | dissipative part of \hat{T}_k^f |
| τ_{kl}^f | dissipative part of t_{kl}^f |
| Φ^α | α -phase exchange of entropy due to mechanical interaction, |
| φ^α | α -phase entropy flux vector, |
| ω_{kl}^α | α -phase vorticity tensor; defined in equation (A4). |

Special notation

| | |
|---------------------|---|
| $A_{k(l}B_{m)n}$ | Symmetric part of $A_{kl}B_{mn}$ with respect to the indices l and m , defined in equation (20c), |
| $A_{k(l}B_{m)l}$ | anti-symmetric part of $A_{kl}B_{mn}$ with respect to the indices l and m ; defined in equation (A4), |
| $\frac{D}{Dt}$ | material derivative of a function with respect to the α -phase velocity, defined in equation (14), |
| D/Dt | material derivative of a function with respect to the system average velocity; defined in equation (18), |
| $\sum_{\alpha=f,s}$ | summation over the fluid and solid phases, |
| $\bar{}$ | the terms preceding the bar must be evaluated at equilibrium, |
| $()_0$ | the terms inside the parenthesis must be evaluated at $Z_\lambda = 0$. |

REFERENCES

- 1 Gray, W. G. and O'Neill, K. On the general equations for flow in porous media and their reduction to Darcy's law, *Water Resour. Res.*, 1976, **12**, 148
- 2 Neuman, S. P. Theoretical derivation of Darcy's law, *Acta Mechanica*, 1976, **25**, 153
- 3 Slattery, J. C. Single-phase flow through porous media, *AIChE J.* 1969, **15**, 866
- 4 Whitaker, S. Advances in theory of fluid motion in porous media, *Ind. Eng. Chem.* 1969, **61**, 14
- 5 Ahmadi, G. and Farshad, M. On the continuum theory of solid-fluid mixtures — a superimposed model of equipresent constituents, *Indian J. Technol.* 1974, **12**, 195
- 6 Kenyon, D. E. Thermostatics of solid-fluid mixtures, *Arch. Rat. Mech. Anal.*, 1976, **62**, 117
- 7 Kenyon, D. E. The theory of an incompressible solid-fluid mixture, *Arch. Rat. Mech. Anal.* 1976, **62**, 131
- 8 Bowen, R. M. Theory of mixtures, *Continuum Physics*, (Ed. A. C. Eringen) 1976, Vol. 3, Part 1, Academic Press, New York
- 9 Crochet, M. J. and Naghdi, P. M. On constitutive equations for flow of fluid through an elastic solid, *Int. J. Eng. Sci.* 1966, **4**, 383
- 10 Bedford, A. and Ingram, J. D. A continuum theory of fluid saturated porous media, *J. Appl. Mech.* 1971, **38**, 1
- 11 Mokadam, R. G. Thermodynamic analysis of Darcy's law, *Trans. ASME, J. Appl. Mech.* 1961, **83**, 208
- 12 Fulks, W. B., Guenther, R. B. and Boetman, E. L. Equations of motion and continuity for fluid flow in a porous medium, *Acta Mechanica*, 1971, **12**, 121
- 13 Green, A. F. and Naghdi, P. M. The flow of fluid through an elastic solid, *Acta Mechanica* 1970, **9**, 329
- 14 Green, A. E. and Steel, T. R. Constitutive equations for interacting continua, *Int. J. Eng. Anal.* 1966, **4**, 483
- 15 Dunwoody, N. T. A thermodynamic theory of diffusion in solid-fluid mixtures, *Arch. Rat. Mech. Anal.* 1970, **38**, 348
- 16 Raats, P. A. C. The role of inertia in the hydrodynamics of porous media, *Arch. Rat. Mech. Anal.* 1971, **44**, 267
- 17 Garg, S. K. Wave propagation effects in a fluid-saturated porous solid, *J. Geophys. Res.* 1971, **76**, 7947
- 18 Garg, S. K., Brownell, Jr. D. H., Pritchett, J. W. and Herrmann, R. G. Shock wave propagation in fluid-saturated porous media, *J. Appl. Phys.* 1975, **46**, 702
- 19 Drew, D. A. and Segel, L. A. Averaged equations for two-phase flows, *Stud. Appl. Math.* 1971, **50**, 205
- 20 Saffman, P. G. On the boundary conditions on the surface of a porous medium, *Stud. Appl. Math.* 1971, **50**, 93
- 21 Bedford, A. and Drumheller, D. S. A variational theory of immiscible mixtures, *Arch. Rat. Mech. Anal.* 1978, p. 37
- 22 Hassanizadeh, M. and Gray, W. G. General conservation equations for multi-phase systems: 1. Averaging technique, *Adv. Water Res.* 1979, **2**, 131
- 23 Hassanizadeh, M. and Gray, W. G. General conservation equations for multi-phase systems: 2. Mass, momenta, energy and entropy equations, *Adv. Water Res.* 1979, **2**, 191
- 24 Coleman, B. D. and Noll, W. Thermodynamics of elastic materials with heat conduction and viscosity, *Arch. Rat. Mech. Anal.* 1963, **13**, 167
- 25 Eringen, A. C. *Mechanics of Continua*, John Wiley, New York 1967
- 26 Muller, I. A thermodynamic theory of mixtures of fluids, *Arch. Rat. Mech. Anal.* 1968, **28**, 1
- 27 Green, A. E. and Naghdi, P. M. Entropy inequalities for mixtures, *Q. J. Mech. Appl. Math.* 1971, **24**, 473
- 28 Halmos, P. R. *Finite Dimensional Vector Spaces*, 2nd edn, van Nostrand, Princeton, 1958
- 29 Avis, R. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice-Hall, Englewood Cliffs, 1962