



A Theoretical Model of Hysteresis and Dynamic Effects in the Capillary Relation for Two-phase Flow in Porous Media

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Abstract. It is well known that the relationship between capillary pressure and saturation, in two-phase flow problems demonstrates memory effects and, in particular, hysteresis. Explicit representation of full hysteresis with a myriad of scanning curves in models of multiphase flow has been a difficult problem. A second complication relates to the fact that P^c – S relationships, determined under static conditions, are not necessarily valid in dynamics. There exist P^c – S relationships which take into account dynamic effects. But the combination of hysteretic and dynamic effects in the capillary relationship has not been considered yet. In this paper, we have developed new models of capillary hysteresis which also include dynamic effects. In doing so, thermodynamic considerations are employed to ensure the admissibility of the new relationships. The simplest model is constructed around main imbibition and drainage curves and assumes that all scanning curves are vertical lines. The dynamic effect is taken into account by introducing a damping coefficient in P^c – S equation. A second-order model of hysteresis with inclined scanning curves is also developed. The simplest version of proposed models is applied to two-phase incompressible flow and an example problem is solved.

Key words: Two-phase flow, capillary pressure, hysteresis, capillary dynamic effects, thermodynamics of porous media.

1. Introduction

The following equations are usually employed to describe two-phase immiscible flow in porous media (see e.g. Collins, 1961; Bear *et al.*, 1968):

$$\frac{\partial}{\partial t}(\varepsilon^\alpha \rho^\alpha) + \nabla \cdot (\rho^\alpha \mathbf{q}^\alpha) = 0, \quad (1)$$

$$\mathbf{q}^\alpha = \frac{K_0 K^\alpha(S)}{\mu^\alpha} (-\nabla p^\alpha + \rho^\alpha \mathbf{g}), \quad (2)$$

$$p^n - p^w = P^c(S). \quad (3)$$

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This system consists of two mass balance equations, Darcy relations for the two phases, and capillary condition. Here the index α designates either wetting phase ($\alpha = w$) or non-wetting phase ($\alpha = n$), and ε^α stands for volume fraction, so that $\varepsilon = \varepsilon^w + \varepsilon^n$ is porosity. For current work, we are not considering systems with mixed wettability. The vector fields \mathbf{q}^α are volumetric fluxes of the fluids, and \mathbf{g} is the gravity vector. The symbol $S = \varepsilon^w/\varepsilon$ stands for the saturation of the wetting phase ($0 \leq S \leq 1$), p^α and ρ^α are pressures and densities of the phases, respectively. In the case of compressible flow, the densities are given as functions of pressures, and $p^\alpha = p^\alpha(\mathbf{x}, t)$ together with saturation $S(\mathbf{x}, t)$ are the functions to be found. For incompressible flows, the densities are given constants. In these equations, all quantities are at the macroscopic scale. They are averaged over some representative elementary volume (REV). Under non-isothermal conditions, system (1)–(3) should be supplemented by heat equation.

The functions $K^\alpha(S)$ in Darcy's law (2) are the relative permeabilities of the medium. An important property of these functions is the degeneration of K^w or K^n for low or high values of saturation, correspondingly. Actually, the bounds of saturation are $0 < S_0^w \leq S \leq S_0^n < 1$ where S_0^α are the so-called irreducible saturations. The system of governing equations (1)–(3) does not describe processes outside this interval.

Equation (3) is simply an empirical relationship. It indicates that there is a correlation between the difference in fluid phase pressures, $p^n - p^w$, and the wetting phase saturation. It is merely an assumption that the function $P^c(S)$ is capillary pressure. This association of $p^n - p^w$ with capillary pressure suggests that the difference $p^n - p^w$ is only caused by capillary effects. This is not necessarily correct. There may be contributions to $p^n - p^w$ from viscous and/or gravity effects. In fact, it is well known that the relationship between $p^n - p^w$ and S is not unique and depends on the history of wetting and drying in the porous medium, and the graphs of 'capillary pressure' show significant hysteresis (see e.g. Collins, 1961; Bear *et al.*, 1968). Typical plots of measured $p^n - p^w$ versus S are given in Figure 1.

In general, the hysteresis is characterized by a main drainage curve, $P_{dr}^c(S)$, and a main imbibition curve, $P_{im}^c(S)$. Moreover, a myriad of scanning curves form the loci of intermediate states. In other words, in the plane $p^n - p^w$ versus S , any point of the domain bounded by two curves, $p^n - p^w = P_{im}^c(S)$ and $p^n - p^w = P_{dr}^c(S)$, may correspond to an equilibrium state of the two liquids in porous medium. There is also observed hysteresis in relationships between relative permeabilities and saturation. But that is usually small (Collins, 1961).

In addition to static memory effects (hysteresis), there exist dynamic memory effects. Due to these effects, the trajectories of $(p^n - p^w) - S$ points measured under dynamic flow conditions are not bounded by main drainage and imbibition curves. In fact, there is convincing experimental evidence that $p^n - p^w$ depends on the rate of saturation \dot{S} (e.g. Topp *et al.*, 1967; Smiles *et al.*, 1971; Vachaud *et al.*, 1972; Stauffer, 1978; Hassanizadeh, 1997). In these papers, measured $(p^n - p^w) -$

S points fall above drainage curve or below imbibition curve. Obviously, the simple $(p^n - p^w) - S$ relationship of the current theories is not applicable to these points.

In some particular cases, it is possible to use relation (3) even if the material system demonstrates hysteresis. For instance, in slow processes with monotonically decreasing (or increasing) saturation, the difference in pressures is also a monotone function of time, and this implies the existence of a single-valued relation between $p^n - p^w$ and S . Hence, if one can *a priori* identify regions of monotonous drainage or imbibition, then every region can be endowed with its own capillary function provided the starting state is known. If, however, the drainage and imbibition regimes follow one another, then there is no unique function that could describe capillary processes. In general, both zones of drainage and imbibition are present in a porous medium simultaneously, and the shape of these zones as well as their time evolution are not known *a priori*. They should rather be found, together with other sought variables, from the solution of a mathematical problem than assumed in advance. In this respect, the relation (3) is unsuitable. It must be emphasized that our development is carried at the macroscale, and no homogenization of the pore-scale description is performed here.

Differential equations with hysteresis have been the subject of studies, from the mathematical point of view, since 1960s (see Krasnosel'skiĭ and Pokrovskiĭ, 1989; Visintin, 1994; and references there). Applications are found in plasticity, ferromagnetism and phase transitions theory. Relatively few mathematical papers deal with the capillary hysteresis in the theory of flow in porous media. As for papers on hydrology, neither the main concepts nor rigorous mathematical results of hysteresis theory have received sufficient attention. Some models of capillary hysteresis were developed by Néel (1942, 1943) and Mualem (1973, 1974). But these are not suitable alternatives to capillary relation (3) because the corresponding initial value problem would require, in general, an infinite number of initial conditions. However, in particular cases, if, for instance, the change in saturation is monotonic, these models can be successfully applied to predict the behavior of capillary curves.

The goal of this paper is to develop a theory that provides a closed-form relationship between $(p^n - p^w)$ and S under all conditions, including both the capillary hysteresis and dynamic effects. In doing so, we have used thermodynamic results obtained previously among others by Hassanizadeh and Gray (1990, 1993) and Gray and Hassanizadeh (1991, 1998). In Section 2, a brief description of their approach and relevant results are given. We then impose additional thermodynamic assumptions. Depending on the level of restrictions imposed, different admissible models of the capillary relation are obtained.

In Section 3, we introduce a simple model of the capillary relation that takes into account both hysteresis and dynamic effects. We prove that, in this case, capillary hysteresis with vertical reversible scanning curves is the only admissible model. The properties of the proposed model are explored in Section 4 by solving a one-dimensional test problem. Under some simplifying assumptions on the shape of

constitutive functions, the redistribution of two liquids in a horizontal column has been considered with an initial saturation in the form of a step function, and an explicit solution is obtained.

In Section 5, we consider an example of a second-order model for capillary hysteresis with a more realistic behavior of scanning curves. The corresponding thermodynamic analysis is performed, and necessary assumptions on free energy of the system are formulated in terms of general thermodynamics of porous media. The representation of this model is based on ideas borrowed from plasticity theory where the stress–strain relationships demonstrate similar hysteretic behavior. In the approach presented here, no direct link between model coefficients and microscale properties is obtained. These coefficients have to be evaluated from experimental studies. Also, the range of validity of first-order and second-order models derived here has to be determined from experimental evidence.

2. Capillary Pressure and Thermodynamics

In the foregoing presentation, $P^c(S)$ and $p^n - p^w$ are both called capillary pressure as commonly done in the literature. But, given the fact that even at equilibrium, $p^n - p^w$ depends on the history of flow processes and has no unique relationship with saturation, it is desirable, and even necessary, to have a definition of capillary pressure, P^c , independent of $p^n - p^w$. Of course, such a definition should not be introduced in a heuristic and arbitrary fashion, but must be developed in the framework of a physically-based theory. One such definition has been suggested by Hassanizadeh and Gray (1990) where the capillary pressure is defined as a thermodynamic parameter, and Equation (3) was given not as a definition of $P^c(S)$ but as a constitutive approximation only valid at equilibrium. In any case, capillary equation (3) has to be modified in order to describe hysteretic and dynamic behavior of the system.

Recent advances in thermodynamic theories of multiphase flow in porous media provide a theoretical framework for a proper definition of capillary pressure. In particular, in the work of Hassanizadeh and Gray (1990) on two-phase flow, pressures of fluid phases and capillary pressure are defined in terms of the free energy functions of the phases. In this theory, Helmholtz free energy functions for the phases and interfaces are hypothesized to depend on state variables such as mass density, temperature, saturation, porosity, specific interfacial area, and the solid phase strain tensor. They employ conservation laws for mass, momentum and energy, and the second law of thermodynamics in order to develop constitutive equations which describe two-phase flow in a porous medium. As a result, the following relationship for capillary pressure is obtained:

$$P^c = -S \frac{\partial A^w}{\partial S} - (1 - S) \frac{\partial A^n}{\partial S} - \frac{1}{\varepsilon} \sum_{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial S}, \quad (4)$$

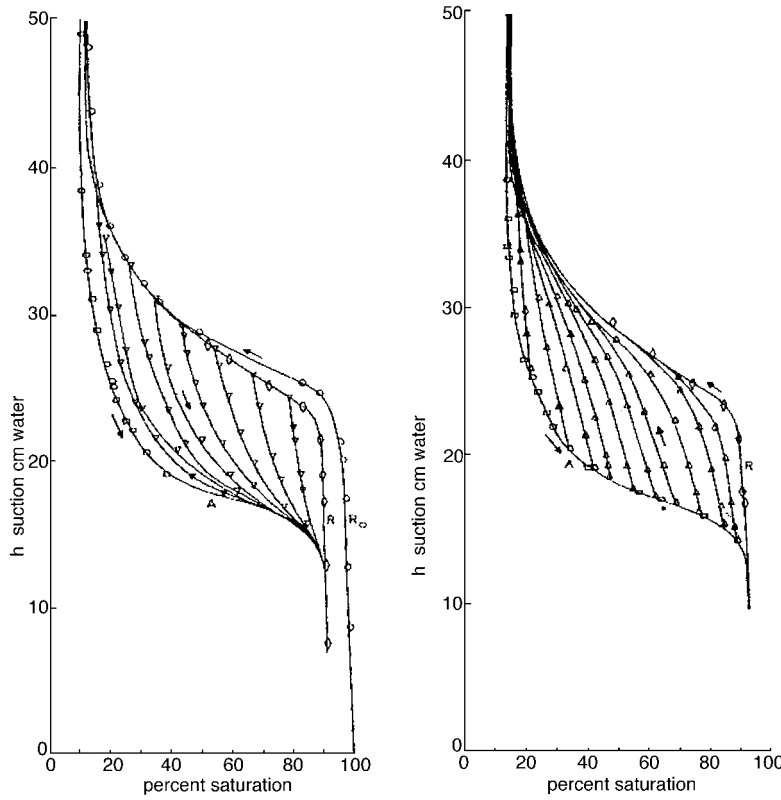


Figure 1. Drainage, imbibition and scanning curves due to Morrow and Harris (1965).

where A^α is the Helmholtz free energy of phase per unit volume of the phase, $A^{\alpha\beta}$ is the Helmholtz free energy of interface per unit volume of the porous medium, and \sum denotes summation over all interfaces. In two-phase porous flow, the superscript $\alpha\beta$ may stand for wn (wetting–nonwetting), ws (wetting–solid) and ns (nonwetting–solid) interfaces. Obviously, the capillary pressure, as defined here, may depend on state variables such as mass density, temperature, porosity and specific interfacial area as well as saturation.

According to Definition (4), capillary pressure is equal to the change in the free energy of the system as a result of a change in the saturation of the wetting phase. Equation (4) explains the spontaneity of an imbibition process. According to this equation, and given the fact that the capillary pressure is always positive, the free energy of the system, that is, $\epsilon SA^w + \epsilon(1 - S)A^n + \sum A^{\alpha\beta}$, must decrease for the saturation of the wetting phase to increase (i.e. for $\Delta S > 0$); this would be a spontaneous process. On the other hand, to decrease the wetting phase saturation ($\Delta S < 0$), the energy of the system must be increased. This will not be a spontaneous process and it will be possible only if external work is performed. In this approach, macroscopic capillary pressure is thus defined solely as an intrinsic

property of the system and is not simply equal to the difference in fluid phases pressures, as is commonly defined. In fact, the difference between P^c and $p^n - p^w$ at a given point in the medium determines whether drainage or imbibition takes place. Such a process will be accompanied with a saturation change and results in an increase in the entropy of the system. Indeed, Hassanizadeh and Gray (1990) show that the following combination of terms contributes to the entropy production:

$$\dots + \varepsilon \dot{S}[P^c - p^n + p^w] - \sum_{\alpha\beta} \dot{a}^{\alpha\beta} \frac{\partial A^{\alpha\beta}}{\partial a^{\alpha\beta}} + \dots \geq 0, \quad (5)$$

where \dot{S} is the temporal derivative of the wetting phase saturation and $a^{\alpha\beta}$ is the total area of $\alpha\beta$ -interface per unit volume of porous medium. Note that in this inequality, \dot{S} and $[P^c - p^n + p^w]$ play the role of a flux and the corresponding force, respectively. Other terms in (5) (some not shown here) can also be interpreted as products of generalized forces and fluxes. The entropy production of the system must be non-negative at any point of the medium for all admissible initial data, boundary conditions and external supplies. Thus, it provides essential restrictions for constitutive relationships between variables in the system. Those restrictions could be used for formulation and classification of various models in continuum mechanics. This idea was suggested by Coleman and Noll (1963). Hassanizadeh and Gray (1980, 1990) and Gray and Hassanizadeh (1991, 1998) developed this method for thermodynamic analysis of processes in porous media.

It can be proven that inequality (5) requires that if $p^n - p^w > P^c$ (for example, because of a decrease in p^w and/or an increase in p^n), then \dot{S} must be negative. This corresponds to drainage taking place. On the other hand, if $p^n - p^w < P^c$, then \dot{S} must be positive; this corresponds to imbibition taking place. Only at equilibrium, when $\dot{S} = 0$ and no change in saturation is occurring, $p^n - p^w$ will be equal to P^c . Under non-equilibrium conditions, \dot{S} will depend on the difference $p^n - p^w - P^c$. These considerations suggest that, in general, one may have

$$\dot{S} = F(P^c - p^n + p^w) \quad \text{or} \quad P^c - p^n + p^w = F^{-1}(\dot{S}) \quad (6)$$

such that

$$\dots + \varepsilon \dot{S} F^{-1} + \dots \geq 0. \quad (7)$$

The traditional form of capillary relation (3) corresponds to the choice $F^{-1}(\dot{S}) \equiv 0$ in which case the first equation in (6) becomes meaningless because F cannot be defined. This choice implies that the interfaces within pores are always in equilibrium with pressures in bulk phases. Strictly speaking, this will occur only under steady-state conditions. Other forms of the functions F^{-1} or F take into account the dynamics of the interfaces. Note that both of these functions may also depend on saturation and other state variables.

In general, relationships between thermodynamic forces and fluxes contain coupling terms, so the temporal derivative of saturation, \dot{S} , may depend on other forces

which contribute to entropy production (5). For simplicity, all coupling effects are neglected in (6). Later on, in Section 5, this assumption will be partially relaxed.

For a linear theory, Hassanizadeh and Gray (1990) have suggested the following approximate relationship:

$$p^n - p^w - P^c + L^c \dot{S} = 0 \tag{8}$$

which corresponds to the following expressions for F and F^{-1} :

$$F = (P^c - p^n + p^w)/L^c, \quad F^{-1} = L^c \dot{S}, \tag{9}$$

where L^c is a material coefficient. Substitution of the second of (9) in (7) reveals that L^c must be non-negative. Thus, the plot of \dot{S} as a function of $p^n - p^w$ is a monotonically decreasing line. A typical graph is shown in Figure 3. In this figure, and in the rest of this manuscript, it is assumed that L^c is a constant. This assumption has no significant influence on the results presented here.

Equation (8) suggests that if we were to plot $p^n - p^w$ as a function of saturation, the resulting curve would also depend on \dot{S} . This is the dynamic effect which was mentioned in the previous section. The value of L^c may be evaluated in an approximate manner from experimental studies reported in the literature. For example, consider experimental graph reproduced in Figure 2 from the paper by Topp *et al.* (1967). One may consider curve A to represent the static capillary pressure P^c . Curve B is considered to be the dynamic capillary pressure $p^n - p^w$. Then, according to Equation (8), the difference $(p^n - p^w) - P^c$ is equal to $-L^c \dot{S}$. The mean value of the term \dot{S} is evaluated from the time it takes for a given change in saturation to occur. For example, for curve B, the water content changes from 0.36 (saturated) to 0.09 in 100 min. This corresponds to an average value of

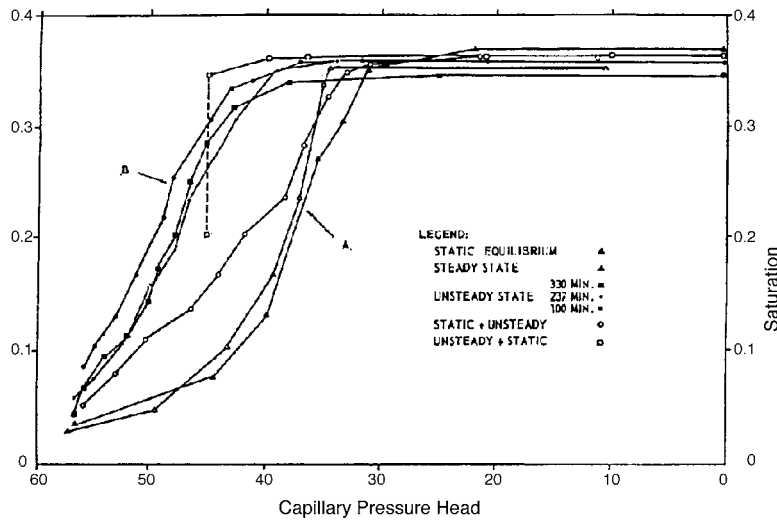


Figure 2. Static capillary pressure and $p^n - p^w$ curves, due to Topp *et al.* (1967).

$1.25 \times 10^{-4} \text{ s}^{-1}$ for \dot{S} . With these estimates, L^c may be calculated. An extensive evaluation of the dynamic capillary coefficient L^c from published works has been carried out by Hassanizadeh (1997). For drainage processes in sandy soils, L^c was found to be in the interval $3 \times 10^4 - 5 \times 10^7 \text{ kg m}^{-1} \text{ s}^{-1}$.

An equation similar to (8) has been obtained by Panfilov (1998) for the megascopic (field) scale in the framework of homogenization theory. He considered system (1)–(3) in a porous medium with small-scale heterogeneities. He then obtained (8) by means of asymptotic homogenization procedure, for an equivalent homogeneous porous medium at the megascale.

Equation (8) describes memory effects in the relationship S versus $p^n - p^w$ because at any instant $t > 0$, the saturation $S(t)$ depends on the evolution of pressure difference at all preceding instances. Considering (8) as an ordinary differential equation with respect to $S(t)$, we could express its solution in the form $S(t) = S\{S(0); p^n(\tau) - p^w(\tau), 0 \leq \tau \leq t\}$. However, this operator does not take into account the phenomenon of hysteresis. Indeed, for slow processes, the memory term $L^c \cdot \dot{S}$ vanishes, and this operator degenerates into the traditional relation (3) without memory, whereas hysteresis is believed to be a rate-independent memory effect (cf. Flügge *et al.*, 1965). Therefore, even in the limit of infinitely slow processes, the memory dependence should be present in capillary relation with hysteresis. Non-local-in-time operators with this property are called hysteresis operators. (cf. Visintin, 1994). In the next section, Equation (8) is modified in order to include a simple form of hysteresis.

As mentioned above, Equation (4) suggests that the capillary pressure depends not only on saturation but also on specific interfacial areas of interfaces. In fact, according to Hassanizadeh and Gray (1993), the dependence on specific interfacial area of fluid–fluid interfaces may (partially) explain the hysteresis in capillary pressure–saturation relationship. Some encouraging results in this regard have recently been obtained by Reeves and Celia (1996). The difficulty with the inclusion of interfacial areas, however, is that their measurement at the macroscale is a formidable task. Moreover, because a new unknown is introduced, new equations, additional to mass balance and Darcy's law, will be needed. These equations will contain new coefficients whose identification and evaluation are beyond current experimental means. The novel approach presented here for the inclusion of both hysteresis and dynamic effects, however, makes use of results of traditional $P^c - S$ measurements.

3. A Simple Model of Capillary Hysteresis

In order to construct a mathematical model of a capillary relation with hysteresis, we need to make assumptions on the dependencies of Helmholtz free energy functions and, as a result, P^c . As a first approximation, we assume that a phase free energy is a function of its mass density only, and that interfacial free energies are functions of saturation only. That is, dependency of free energy functions on spe-

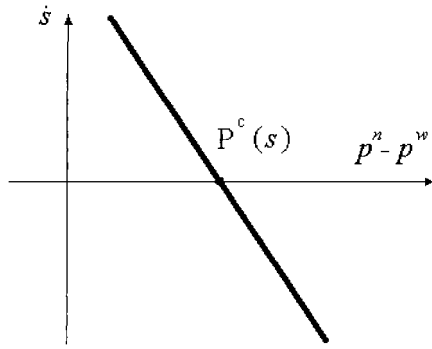


Figure 3. A relation between \dot{S} and $p^n - p^w$ without hysteresis.

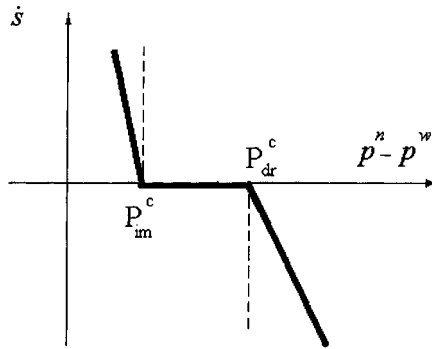


Figure 4. Relation between \dot{S} and $p^n - p^w$ with hysteresis.

cific interfacial areas is neglected. Thus, the following form of the total free energy of the system per unit volume of the porous medium under isothermal conditions is assumed:

$$A_{\text{tot}} = \varepsilon^n A^n(\rho^n) + \varepsilon^w A^w(\rho^w) + \sum_{\alpha\beta} A^{\alpha\beta}(S). \tag{10}$$

With these assumed dependencies, and from Equation (4), one finds that P^c is a function of saturation, S , only. If we assume that terms pertaining to temporal derivatives of interfacial areas are negligible, then equalities (6) are the only candidates for capillary relation. Nevertheless, even within these restrictive circumstances, it is possible to include hysteresis in the capillary relation in a simple way.

Without hysteresis, the plot of relation (6) in the plane of \dot{S} versus $p^n - p^w$ is a strictly decreasing curve as shown in Figure 3 (we get a straight line for linear approximation (8)). Here it is assumed that $p^n - p^w$ has a unique value $P^c(S)$ at $\dot{S} = 0$. However, with hysteresis included, the expression $P^c - p^n + p^w$ in Equation (6) will not be a single-valued function of \dot{S} . In this case, the plot of relation (6) has a flat zone as it is shown in Figure 4. That is, if $P_{\text{im}}^c(S)$ and $P_{\text{dr}}^c(S)$ are considered as the bounding curves of equilibrium states, all points between the

curves $p^n - p^w = P_{\text{dr}}^c(S)$ and $p^n - p^w = P_{\text{im}}^c(S)$ on the plane $p^n - p^w$ versus S are admissible states of the system at $\dot{S} = 0$. Then, the inverse function $F^{-1}(\dot{S})$ is multi-valued at $\dot{S} = 0$. For the linear case, the following explicit expression can be provided:

$$\dot{S} = F := \begin{cases} (P_{\text{im}}^c(S) - p^n + p^w)/L_{\text{im}}^c & \text{if } P_{\text{im}}^c(S) - p^n + p^w \geq 0 \\ 0 & \text{if } P_{\text{im}}^c(S) \leq p^n - p^w \leq P_{\text{dr}}^c(S) \\ (P_{\text{dr}}^c(S) - p^n + p^w)/L_{\text{dr}}^c & \text{if } P_{\text{dr}}^c(S) - p^n + p^w \leq 0. \end{cases} \quad (11)$$

To complete the definition of F , the dynamic capillary coefficients L_{im}^c and L_{dr}^c should be measured or defined, and also the two main curves $P_{\text{dr}}^c(S)$ and $P_{\text{im}}^c(S)$ should be prescribed.

It is worth noting that, although the thermodynamic ‘force’ $P^c - p^n + p^w$ can not be expressed as a single-valued function of \dot{S} , with the function F defined by formula (11), the function $\dot{S}F^{-1}(\dot{S})$ is single valued. Thus, it is possible to express the corresponding part of entropy production in terms of \dot{S} . The explicit formula reads

$$\dot{S}F^{-1}(\dot{S}) = \begin{cases} (P^c(S) - P_{\text{im}}^c(S))\dot{S} + L_{\text{im}}^c\dot{S}^2 & \text{if } \dot{S} \geq 0 \\ (P^c(S) - P_{\text{dr}}^c(S))\dot{S} + L_{\text{dr}}^c\dot{S}^2 & \text{if } \dot{S} \leq 0 \end{cases}. \quad (12)$$

It is apparent that the entropy production is not a differentiable function of \dot{S} . It has a conic singularity at the point $\dot{S} = 0$, in contrast with situations without hysteresis.

The second law of thermodynamics in the form of inequality (7) provides the following restrictions for parameters of this expression:

$$L_{\text{dr}}^c \geq 0, \quad L_{\text{im}}^c \geq 0, \quad P_{\text{im}}^c(S) \leq P^c(S) \leq P_{\text{dr}}^c(S). \quad (13)$$

Note that the capillary pressure $P^c(S)$, defined by Equation (4), is considered to be a unique function of saturation. However, it does not appear in expression (11) for the function F . Therefore, the last restriction is not of any direct use here.

The most important feature of the function F is its uniformly zero value in the interval $P_{\text{im}}^c(S) \leq p^n - p^w \leq P_{\text{dr}}^c(S)$. This and only this property provides hysteresis in the capillary relation. The linear behavior of F outside that interval is not too essential to our hysteresis model and is assumed here as an approximation. The only reason for the linearity is the wish to minimize the number of parameters to be measured in experiments. It should be noted that the differential equation (11) defines an operator $(p^n - p^w) \rightarrow S$ with memory. This operator is rate dependent, and the memory dependence does not disappear in the limit case of infinitely slow processes as it takes place for non-hysteretic relation (8). Hence, relation (11) describes a hysteresis coupled with viscous-type memory effects.

In order to get ‘pure’ hysteretic properties, we should set $L_{\text{im}}^c = L_{\text{dr}}^c = 0$. Then the function F has to be transformed into multi-valued graph on the plane \dot{S} versus $p^n - p^w$. Equation (11) with this multi-valued F should be re-written in the

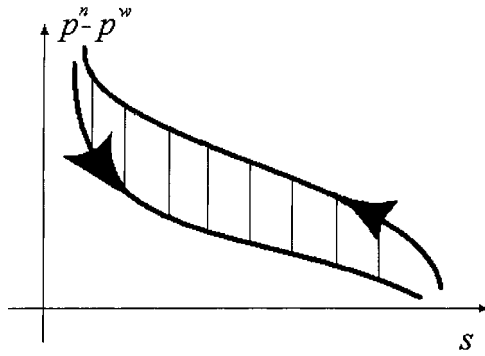


Figure 5. Play-type capillary hysteresis.

form of implication $\dot{S} \in F(S, p^n - p^w)$. Implications of this kind are commonly used in convex analysis and in the mathematical theory of hysteresis. They could be interpreted as a generalization of traditional differential equations. Usually the mathematical analysis and numerical studies of governing equations with multi-valued constitutive relations are more difficult. Therefore, it is desirable to have the relaxation of pure hysteresis, by introduction of non-zero damping coefficients L_{im}^c and L_{dr}^c into capillary relation, even if the dynamic effect in a process under consideration is negligibly small.

Expression (11) implies that $\dot{S} = 0$ for any intermediate process with values $p^n - p^w$ satisfying the inequality $P_{im}^c(S) < p^n - p^w < P_{dr}^c(S)$. This means that the transition from drainage to imbibition states, and vice versa, occurs at a constant saturation. On the plane $p^n - p^w$ versus S , the processes of this kind are represented by vertical straight lines, as shown in Figure 5. The model of hysteresis with reversible vertical scanning curves has many applications in physics. In the mathematical theory of hysteresis, it is called play-type hysteresis (see Visintin (1994)). In practice, the family of scanning curves is not a set of vertical reversible straight lines. Thus the above simple form of capillary hysteresis can be considered as a rough approximation of real processes. Note that this simple model is a direct consequence of assumption (10) on the dependencies of free energy of the system. In fact, play-type hysteresis is the only model consistent with this thermodynamic assumption. All other models would imply more complicated dependencies of free energy on system properties.

The closed system of equations with the proposed form of capillary hysteresis, for incompressible fluids, reads

$$\varepsilon \dot{S} = \nabla \cdot \left(\frac{K_0 K^w(S)}{\mu^w} (\nabla p^w - \rho^w \mathbf{g}) \right), \tag{14}$$

$$-\varepsilon \dot{S} = \nabla \cdot \left(\frac{K_0 K^n(S)}{\mu^n} (\nabla p^n - \rho^n \mathbf{g}) \right), \tag{15}$$

$$\begin{aligned} \dot{S} = & \frac{1}{2L_{\text{im}}^c} [(P_{\text{im}}^c(S) - p^n + p^w) + |P_{\text{im}}^c(S) - p^n + p^w|] + \\ & + \frac{1}{2L_{\text{dr}}^c} [(P_{\text{dr}}^c(S) - p^n + p^w) - |P_{\text{dr}}^c(S) - p^n + p^w|]. \end{aligned} \quad (16)$$

Here, equality (16) for the capillary relation is the only innovation to traditional equations (1)–(2). The expression is simply the algebraic form of the piece-wise linear function F shown in Figure 4.

These equations should be supplemented by appropriate boundary and initial conditions. Note that only one initial condition is needed, e.g. initial saturation distribution. This is a definite advantage of the proposed model, which is the result of assumption (10) and vertical scanning curves. All other models of capillary hysteresis imply more complicated thermodynamics of the system and, as a result, require more initial data, which are usually unavailable.

It should be noted that neither free energy A_{tot} nor capillary pressure $P^c(S)$, as defined by (4), appear in Equations (14)–(16). In the framework of the proposed model, they remain undefined and cannot be determined when Equations (14)–(16) are solved. Nevertheless, they are taken here into account implicitly because both the proposed model (Equation (16)) and inequalities (13) for the main drainage and imbibition curves imply certain restrictions for these thermodynamic parameters.

4. Horizontal Redistribution with Capillary Hysteresis

To illustrate properties of our proposed model, we consider the problem of counter-current redistribution of two liquids in a 1D horizontal porous column. The governing equations are (14)–(16) in x -direction, with no gravity. In this problem, let the domain under consideration be the whole axis $-\infty < x < +\infty$, and the initial water saturation S_0 be given in the form of a step function

$$S_0 = \begin{cases} S_+, & x > 0 \\ S_-, & x < 0 \end{cases} \quad (17)$$

with constants S_+ and S_- such that $S_+ > S_-$ and $P_{\text{dr}}^c(S_+) < P_{\text{im}}^c(S_-)$. As boundary conditions we impose zero fluxes at infinity, $\mathbf{q}^\alpha|_{x=\pm\infty} = 0$. To complete the problem definition, we impose continuity conditions at $x = 0$. In this respect, pressures and fluxes are assumed to be continuous functions of x even though there is a jump in saturation.

The peculiarity of this problem is that drainage and imbibition zones can be identified a priori due to the special form of initial data. It can be assumed that saturation S decreases in time everywhere to the right from origin $x = 0$, and it increases for any $x < 0$. Due to this peculiarity, the problem may be decomposed into two adjacent problems, with different non-hysteretic capillary relations, in the domains $x > 0$ and $x < 0$. These two problems are coupled by jump conditions at the point $x = 0$. It should be noted that, in general, drainage and imbibition

zones are not given *a priori*. In this respect, the above problem is an exception. For example, if the same problem is considered in a finite interval or if there is a continuous initial saturation distribution, the decomposition does not necessarily apply.

The above problem has been solved by Philip (1991) in the case of ‘pure’ hysteresis without any dynamic terms. In fact, he considered the redistribution problem in an unsaturated porous medium, but his results can be easily extended to two-phase flow. Also, he did not specify a model of capillary hysteresis because only two capillary curves were needed in his solution and, thus, the shapes of scanning curves were insignificant. Philip’s solution possesses a similarity property because the problem is invariant with respect to transformations $x \rightarrow \xi x, t \rightarrow \xi^2 t$ for any $\xi > 0$. The solution was given in an implicit form which could be analyzed qualitatively. If $P_{dr}^c(S_+) \geq P_{im}^c(S_-)$, then the solution of Philip is trivial and does not depend on time at all. In this case, the pressures are arbitrary constants such that the value of $p^n - p^w$ lies between $P_{dr}^c(S_+)$ and $P_{im}^c(S_-)$. Otherwise, the solution is not trivial and behaves as follows. For $x \neq 0$ and $t > 0$, the solution $S(x, t)$ is continuous and depends on x/\sqrt{t} only. Its left and right limits at the origin are S_l and S_r , with $S_- < S_l < S_r < S_+$, such that $P_{dr}^c(S_r) = P_{im}^c(S_l)$. Thus, the magnitude of the initial jump ($S_+ - S_-$) instantly drops down to the value $(S_r - S_l)$, and remains fixed thereafter.

The similarity solution does not hold for Equations (14)–(16) where the coefficients L_{dr}^c and L_{im}^c are not zero. In order to get an explicit solution, we consider a simplified version of this problem and impose the following restrictions

$$L_{dr}^c = L_{im}^c = L, \quad \frac{K_0 K^\alpha(S)}{\varepsilon \mu^\alpha} = K_\alpha, \tag{18}$$

$$P_{dr}^c(S) = P_0 - bS + \sigma, \quad P_{im}^c(S) = P_0 - bS - \sigma,$$

where L, b, K_α, P_0 and σ are assumed to be positive constants independent of S . This assumption is of course not met for real systems and it is made here only to make it possible to obtain an explicit analytical solution to this problem. As far as properties of the analytical solution is concerned, this assumption has no significant consequences. That is, the trend presented in Figure 6 will remain valid. For application to a real system, usual dependence of relative permeability (and other parameters) on saturation should be taken into account. Note that for the case that there is no hysteresis, we will have $\sigma = 0$.

With these assumptions, the following closed-form solution can be obtained (the details of derivation are given in the Appendix):

$$S(x, t) = \frac{S_+ + S_-}{2} + \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(kx)}{k} \times$$

$$\times \left\{ \left(\frac{S_+ - S_-}{2} - \frac{\sigma}{b} \right) \exp\left(-\frac{b\omega^2 k^2 t}{L(1 + \omega^2 k^2)} \right) + \frac{\sigma}{b} \right\} dk, \tag{19}$$

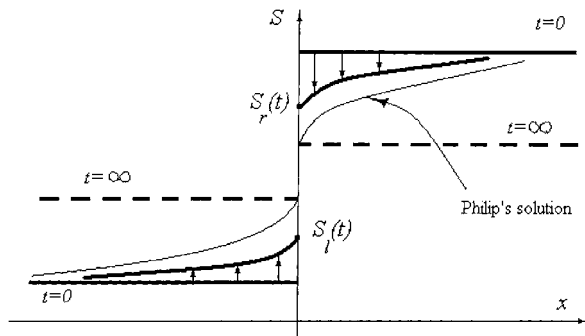


Figure 6. Analytical solution of redistribution problem in infinite horizontal column including dynamic effects.

with

$$\omega = \sqrt{\frac{K_n K_w L}{K_n + K_w}}$$

Note that, in order to check whether this expression satisfies initial data (17), one needs to use the identity

$$\frac{2}{\pi} \int_0^\infty \frac{\sin(kx)}{k} dk = \text{sign}(x). \tag{20}$$

As $t \rightarrow \infty$, the exponential term in (19) goes to zero. Thus, due to (20), the large-time equilibrium profile of saturation is a step function with magnitude $2\sigma/b$. In Philip's solution, the large-time equilibrium profile is the same.

Let us investigate the behavior of the solution in the vicinity of the origin $x = 0$. The integral in (19) is conditionally convergent, so we can not evaluate this formula at $x = 0$. Let's decompose the exponent in (19) as follows:

$$\begin{aligned} & \exp\left(-\frac{b\omega^2 k^2 t}{L(1 + \omega^2 k^2)}\right) \\ &= \exp\left(-\frac{bt}{L}\right) + \left[\exp\left(-\frac{b\omega^2 k^2 t}{L(1 + \omega^2 k^2)}\right) - \exp\left(-\frac{bt}{L}\right) \right]. \end{aligned}$$

The second term here goes to zero as $k \rightarrow +\infty$, and thus, the corresponding part of the integral in (19) is absolutely convergent to some odd continuous function of x . The first term does not depend on k , and the corresponding integral in (19) provides an explicit expression with jump-type singularity due to identity (20).

Thus, we have

$$\begin{aligned} S_l(t) &:= S(-0, t) = \frac{S_+ + S_-}{2} - \frac{\sigma}{b} - \left(\frac{S_+ - S_-}{2} - \frac{\sigma}{b} \right) \exp \left\{ -\frac{b}{L}t \right\}, \\ S_r(t) &:= S(+0, t) = \frac{S_+ + S_-}{2} + \frac{\sigma}{b} + \left(\frac{S_+ - S_-}{2} - \frac{\sigma}{b} \right) \exp \left\{ -\frac{b}{L}t \right\}, \\ [S]_0(t) &:= S_r(t) - S_l(t) = 2\frac{\sigma}{b} + 2 \left(\frac{S_+ - S_-}{2} - \frac{\sigma}{b} \right) \exp \left\{ -\frac{b}{L}t \right\}. \end{aligned}$$

So, the magnitude of initial jump drops in time from $S_+ - S_-$ to the limit value $2\sigma/b$. This is in contrast with the case of pure capillary hysteresis, where this drop occurs instantaneously.

The qualitative behavior of the solution is shown in Figure 6. Comparing graphs of $S(x, t)$ with Philip's solution, one can see that our model provides a delay in time which is caused by dynamic effects. The typical value of relaxation time is L/b or, in terms of P^c -functions, $L(-dP^c/dS)^{-1}$.

5. A Second-order Model of Capillary Hysteresis

In the simple model of capillary hysteresis presented in Section 3, the transition from imbibition to drainage, and vice versa, occurs at a constant saturation. In other words, the hysteresis scanning curves are vertical lines as illustrated in Figure 5. In real systems, however, the intermediate processes are accompanied by variation of saturation. In this section, we generalize our simple hysteresis model in order to allow for inclined scanning curves. This may be considered as a second-order model of capillary hysteresis.

First, we set restrictions on the behavior of the scanning curves in the plane $p^n - p^w$ versus S . We assume that they are inclined reversible curves. That is, the trajectories of processes on the plane $p^n - p^w$ versus S from drainage to imbibition, and vice versa, are assumed to coincide. A schematic diagram of this hysteretic rule is shown in Figure 7. In the mathematical theory, this is called Prandtl–Ishlinsky model of hysteresis. It is widely applied in plasticity for the modeling of hysteresis in stress–strain relationships (see Hill, 1950).

We are aware that this second order model is still a strong simplification of real systems. For example, Morrow and Harris (1965) obtained water–air equilibrium curves for an artificial porous material of glass beads. In Figure 7, drainage, imbibition and intermediate curves are shown. From these graphs, it is evident that the behavior of the intermediate curves in Figure 1 is not quite the same as assumed in Figure 7. Scanning curves always approach the main drainage or imbibition curves asymptotically. This effect is neglected in the model of scanning curves shown in Figure 7. Some models of hysteresis were suggested by Néel (1942, 1943) and developed by Mualem (1973, 1974). They take into account the asymptotic behavior of scanning curves but the corresponding equations contain infinite number of unknowns. We do not consider this kind of hysteresis here.

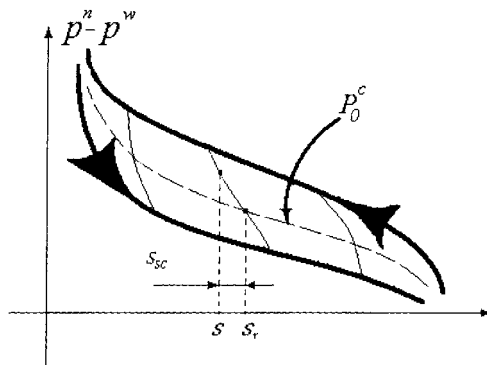


Figure 7. Prandtle–Ishlinsky model of hysteresis.

In Prandtle–Ishlinsky model, if the initial saturation and pressure distribution is given and the sense of evolution of $p^n - p^w$ is also known, then the values of saturation S at any instance can be determined by means of simple rules implied by Figure 7. Formally, this figure defines the following operator

$$S(t) = S\{S(0); p^n(\tau) - p^w(\tau), 0 \leq \tau \leq t\}, \quad (21)$$

if the two main capillary functions, $P_{dr}^c(S)$ and $P_{im}^c(S)$, and all scanning curves are known. Formally, this operator could be employed as a capillary relation instead of equality (3) or our simple hysteresis model (11). If, for example, the state of the system corresponds to a point inside the domain between two main capillary curves and the pressure difference $p^n - p^w$ increases, then this point moves upward along the scanning curve to the drainage boundary of the hysteresis zone. Thereafter, it moves along this boundary to the left. As soon as the pressure difference begins to decrease, the point leaves the drainage curve and moves downward along a new scanning curve to the imbibition trajectory, and so on. Operator (21), however, is not suitable for practical applications in graphical form. Instead, it is necessary to represent it by a closed-form expression.

To this end, we introduce new variables in the relationship between $p^n - p^w$ and S . First, let us choose an arbitrary reference curve $p^n - p^w = P_0^c(S)$, somewhere between the main drainage and imbibition trajectories, which intersects all scanning curves. In Figure 7, this curve is shown as a dashed line. For instance, it is possible to set $P_0^c(S) = (P_{im}^c + P_{dr}^c)/2$. Next, consider a point $(p^n - p^w, S)$ on a given scanning curve within the hysteresis loop. Let S_r be the saturation coordinate of the intersection of the scanning curve with reference curve P_0^c . Finally, define a local variable on the scanning curve by $S_{sc} = S - S_r$. The pair of variables (S_r, S_{sc}) can be considered as new coordinates in the plane $p^n - p^w$ versus S . In terms of these variables, each scanning curve satisfies the equation $S_r = \text{const}$, and the reference curve $P_0^c(S)$ corresponds to the equation $S_{sc} \equiv 0$. The local variable S_{sc} is negative for points above P_0^c -curve and positive for points below it. The idea to use decomposition of the total saturation into two parts is borrowed from the theory

of plasticity ([Hi50]) where the strain tensor is usually decomposed into sum of elastic and plastic terms.

The change of variables $(p^n - p^w, S)$ to (S_r, S_{sc}) is given by the following expressions:

$$S = S_r + S_{sc}, \quad (22)$$

$$p^n - p^w = P_{sc}^c(S_r, S_{sc}), \quad (23)$$

where the explicit form of the function P_{sc}^c depends on the particular shape of the scanning curves and on the choice of the reference capillary function $P_0^c(S)$. In the vicinity of reference capillary curve, the variable S_{sc} is small and expression (23) has the form

$$p^n - p^w = P_{sc}^c = P_0^c(S) - E(S)S_{sc} + O(S_{sc}^2), \quad (24)$$

where the coefficient $E(S)$ is the slope of the scanning curve.

The new coordinates are introduced with the help of $P^c - S$ diagram which represents the behavior of equilibrium, that is infinitely slow processes. The equilibrium states on the plane $p^n - p^w$ versus S are bounded by main drainage and imbibition capillary curves. That is why the mapping (22)–(23) is defined in the zone of hysteresis only. In order to take into account dynamic effects, we have to assume that transformation of variables (22)–(23) is extended to outside this zone.

We still need to propose a capillary pressure–saturation relationship such as (6) or (8); actually a relationship between $p^n - p^w$ and S or S_r . We shall do this with guidance from thermodynamic considerations. It was proven in Section 3 that assumption (10), where interfacial free energy depends on saturation only, implies that the play-type capillary hysteresis is the only admissible model. Now, in order to obtain a more complicated model, we have to introduce an additional independent variable. Following Hassanizadeh and Gray (1990), the interfacial area a^{wn} , the amount of surface between liquid phases per unit volume of the medium, is introduced as an additional independent variable. Thus, instead of (10), the following formula for free energy of the system may be assumed:

$$A_{tot} = \varepsilon^n A^n(\rho^n) + \varepsilon^w A^w(\rho^w) + \sum_{\alpha\beta} A^{\alpha\beta}(S, a^{wn}). \quad (25)$$

However, as we argued earlier, the variable a^{wn} is not yet suitable for use in practical applications and it was omitted from the simple model. In Prandtl-Ishlinsky model of hysteresis, however, two variables are needed. Therefore, we propose to employ S_r and S_{sc} instead of S and a^{wn} as independent variables. It must be emphasized that we are not considering the pair (S_r, S_{sc}) to be equivalent to S and a^{wn} . Formally, we are assuming S_r and S_{sc} to be functions of S and a^{wn} . Then, the pair (S_r, S_{sc}) replaces S and a^{wn} in the set of independent variables.

Let $\tilde{A}(S_r, S_{sc})$ be the sum of interfacial energy functions, $\sum A^{\alpha\beta}$. Then, it can be shown that expression (5) for entropy production may be re-written in terms of S_r and S_{sc} in the following form:

$$\begin{aligned} \dots + \varepsilon \dot{S}_r \left[-\frac{1}{\varepsilon} \frac{\partial \tilde{A}}{\partial S_r} - p^n + p^w \right] + \\ + \varepsilon \dot{S}_{sc} \left[-\frac{1}{\varepsilon} \frac{\partial \tilde{A}}{\partial S_{sc}} - p^n + p^w \right] + \dots \geq 0. \end{aligned} \quad (26)$$

Inside the hysteresis loop, we are always along a scanning curve which means that $\dot{S}_r = 0$, and \dot{S}_{sc} may take upon arbitrary values (positive or negative). Then, in the interior of the hysteresis loop, a sufficient condition for (26) to be valid is:

$$-\frac{1}{\varepsilon} \frac{\partial \tilde{A}}{\partial S_{sc}} - p^n + p^w = 0. \quad (27)$$

This is of course not a necessary condition. One may, instead, assume a relationship between \dot{S}_{sc} and its corresponding flux in (26) such that the inequality is satisfied. That will lead to the introduction of irreversible dynamic effects or even secondary hysteresis loops for the scanning curves. We do not consider these generalizations of the model here, and Equation (27) is employed as the only alternative model.

We assume that the condition (27) holds outside the hysteresis loop as well. As for \dot{S}_r , because it will be negative above the drainage curve and positive below the imbibition curve, we assume it is a function of its corresponding thermodynamic force, $-\varepsilon^{-1} \partial \tilde{A} / \partial S_r - p^n + p^w$. Without hysteresis, the relationship between \dot{S}_r and $p^n - p^w$ will be strictly monotonic, as in Figure 3. However, with hysteresis included, the monotonic relation will have a flat zone, as in Figure 4. Then, according to inequality (26), one must either have: $\dot{S}_r = 0$ (for all intermediate states of the system), or $\dot{S}_r \leq 0$ (for $p^n - p^w \geq P_{dr}^c(S)$), or $\dot{S}_r \geq 0$ (for $p^n - p^w \leq P_{im}^c(S)$). In the case of linear approximation, an equality similar to (11) may be proposed:

$$\dot{S}_r = F(S, p^n - p^w), \quad (28)$$

where

$$F = \begin{cases} (P_{im}^c(S) - p^n + p^w) / L_{im}^c & \text{if } P_{im}^c(S) - p^n + p^w \geq 0 \\ 0 & \text{if } P_{im}^c(S) \leq p^n - p^w \leq P_{dr}^c(S) \\ (P_{dr}^c(S) - p^n + p^w) / L_{dr}^c & \text{if } P_{dr}^c(S) - p^n + p^w \leq 0. \end{cases}$$

Thus, the function F is the same as defined for the simple model. As in the case of play-type hysteresis, expression (28) accounts for dynamic effects.

Substituting relations (27) and (28) into (26), we obtain the following necessary conditions for entropy inequality

$$L_{dr}^c \geq 0, \quad L_{im}^c \geq 0, \quad P_{im}^c(S) \leq -\frac{1}{\varepsilon} \frac{\partial \tilde{A}}{\partial S_r} \leq P_{dr}^c(S) \quad (29)$$

Note that the comparison of (27) with (23) yields:

$$-\frac{1}{\varepsilon} \frac{\partial \tilde{A}}{\partial S_{sc}} = P_{sc}^c(S_r, S_{sc}). \tag{30}$$

In other words, (30) is simply a restatement of (23), here obtained as an equilibrium approximation.

Now, Equation (28) together with relations (22) and (23) replace Equation (3) in the classical theory of two-phase flow. By introducing some more simplifying assumptions, we can make this thermodynamic formalism more useful for practical applications. For example, consider a second order approximation for interfacial energy in the form

$$\tilde{A} = \tilde{A}_0(S) + \frac{\varepsilon}{2} E S_{sc}^2, \tag{31}$$

where E is a constant coefficient, and the first term is supposed to depend on $S = S_r + S_{sc}$ only. Then we get

$$-\frac{1}{\varepsilon} \frac{\partial \tilde{A}}{\partial S_r} = -\frac{1}{\varepsilon} \frac{d\tilde{A}_0}{dS}, \quad -\frac{1}{\varepsilon} \frac{\partial \tilde{A}}{\partial S_{sc}} = P_{sc}^c = -\frac{1}{\varepsilon} \frac{d\tilde{A}_0}{dS} - E S_{sc}. \tag{32}$$

Finally, comparison of (32) and (24) gives

$$-\frac{1}{\varepsilon} \frac{d\tilde{A}_0}{dS} = P_0^c(S).$$

Consequently, the last thermodynamic restriction in (29) holds true automatically if the reference capillary curve, P_0^c , is chosen somewhere between the main drainage and imbibition curves.

In order to specify expressions for capillary curves in this example, let us assume that the main drainage and imbibition curves are given by Brooks and Corey formula:

$$P_{dr}^c(S) = \Pi_{dr} \left(\frac{S - S_0^w}{S_0^n - S_0^w} \right)^{-1/\lambda} \quad P_{im}^c(S) = \Pi_{im} \left(\frac{S - S_0^w}{S_0^n - S_0^w} \right)^{-1/\lambda},$$

where λ , Π_{dr} and Π_{im} are given positive constants. Also, let

$$P_0^c(S) = (P_{dr}^c(S) + P_{im}^c(S))/2. \tag{33}$$

Then equality (23) provides the following explicit equation for scanning curves in the plane of $p^n - p^w$ versus S :

$$p^n - p^w = \frac{\Pi_{dr} + \Pi_{im}}{2} \left(\frac{S - S_0^w}{S_0^n - S_0^w} \right)^{-1/\lambda} - E(S - S_r), \tag{34}$$

where $S_r = \text{const}$ along each scanning curve. This formula indicates that the scanning curves have more or less the same shape as the main imbibition and drainage curves but are corrected by a linear term.

The governing equations of two-phase flow, with Prandtl–Ishlinsky model of capillary hysteresis, are mass balance Equations (14) and (15) and the Equations (22), (28) and (34). Formally, Equations (22) and (34) could be solved in terms of S_r , and this auxiliary variable would be replaced with S and $p^n - p^w$. Then Equation (28) could be represented in terms of difference in pressures $p^n - p^w$, total saturation S and their temporal derivatives. Note that a relationship between $p^n - p^w$ and \dot{S} , in form of equation (6), which worked in the case of play-type hysteresis, is not obtained here.

In the case of approximation (32), the variables S_r and S_{sc} may be written in terms of S and $p^n - p^w$ and the following second-order model of hysteresis is obtained:

$$\left(1 - \frac{1}{E} \frac{dP_0^c(S)}{dS}\right) \dot{S} + \frac{1}{E} (\dot{p}^n - \dot{p}^w) = F(S, p^n - p^w). \quad (35)$$

This equation is a generalization of relation (16) which was the first-order approximation for capillary hysteresis. In the limit, as $E \rightarrow \infty$, this equation reduces to the play-type capillary hysteresis. The additional term inside the parenthesis may be estimated using experimental results of Morrow and Harris (1965). From Figure 1, it follows that, in the saturation range 0.2–0.8, the slope of the mean capillary pressure curve, $-dP_0^c(S)/dS$, is about 7 ± 2 cm H₂O, and the slope of scanning curves, E , is 50–60 cm H₂O. Thus, the contribution of the term $E^{-1}dP_0^c/dS$ in (35) is about 10%. The value of $E^{-1}(\dot{p}^n - \dot{p}^w)$ depends on dynamic behavior of a process, and its contribution should be estimated for each particular problem separately.

Conclusions

The so-called capillary pressure–saturation curves actually represent a correlation between the difference in pressures of fluid phases, $p^n - p^w$, and the wetting phase saturation, S . The correlation is not a unique relationship and contains both static and dynamic memory effects. The static memory effects are manifested in the form of hysteresis in capillary pressure–saturation relationships. The dynamic effects have been observed in the form of dependence of $p^n - p^w$ on the temporal rate of change of saturation, \dot{S} . In this paper, a new model of capillary pressure is developed in the form of an explicit relationship among $p^n - p^w$, S and \dot{S} . A second-order model is also developed which includes a dependence on $\dot{p}^n - \dot{p}^w$. In the derivations of these models, use is made of thermodynamic considerations.

In the simple model, free energies of interfaces are assumed to be functions of saturation only. Hysteresis is represented by two main curves for drainage and imbibition, and all the scanning curves are vertical lines. In the second-order model, hysteresis is also represented by the main drainage and imbibition curves, but the scanning curves have inclination with a given slope. The two models, despite their simplicity, have the potential to account for the major elements of hysteresis and

dynamic effects. The use of the simple model is illustrated by applying it to the redistribution of two liquids in a horizontal infinite column. An analytical solution is obtained and is compared to the solution for the case of traditional equations.

Acknowledgements

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Appendix

Here we give a proof of formula (19) for the solution of 1D redistribution problem in an unbounded horizontal column. Under assumptions (18) for constitutive functions, Equations (14) and (15) are reduced to the following equalities

$$\dot{S} = K_w \frac{\partial^2 p^w}{\partial x^2}, \quad -\dot{S} = K_n \frac{\partial^2 p^n}{\partial x^2}. \quad (36)$$

Summing these equations and integrating, we obtain the relation

$$K_n p^n + K_w p^w = \text{const} := (K_n + K_w) P_{\text{aux}}, \quad (37)$$

where P_{aux} is an arbitrary constant. As done by Philip (1991), we assume that the point $x = 0$ separates the zones of drainage and imbibition for all $t > 0$. Then, from (16), we get the equation

$$L \dot{S} = \begin{cases} P_{\text{dr}}^c(S) - p^n + p^w & \text{if } x > 0 \\ P_{\text{im}}^c(S) - p^n + p^w & \text{if } x < 0. \end{cases} \quad (38)$$

Substitute this expression for \dot{S} in the first equation of (36) and eliminate p^n by means of equality (37). The result is the following ordinary differential equation for p^w :

$$K_n K_w L \frac{\partial^2 p^w}{\partial x^2} = \begin{cases} (K_n + K_w)(p^w - P_{\text{aux}}) + K_n P_{\text{dr}}^c(S) & x > 0 \\ (K_n + K_w)(p^w - P_{\text{aux}}) + K_n P_{\text{im}}^c(S) & x < 0 \end{cases} \quad (39)$$

with

$$[p^w]_0 = [K_w p^w]_0 = 0, \quad x = 0 \quad (40)$$

where $[\cdot]_0$ designates jump of corresponding function. The solution of (39) reads

$$P_{\text{aux}} - p^w = \left\{ \frac{K_n}{K_n + K_w} \int_{-\infty}^{+\infty} P_{\text{dr}}^c(S(\xi)) G(x - \xi) d\xi - C_+ \exp\left(-\frac{x}{\omega}\right) \right\},$$

if $x > 0$, and

$$P_{\text{aux}} - p^w = \left\{ \frac{K_n}{K_n + K_w} \int_{-\infty}^{+\infty} P_{\text{im}}^c(S(\xi)) G(x - \xi) d\xi + C_- \exp\left(+\frac{x}{\omega}\right) \right\},$$

if $x < 0$, where C_+ and C_- are some constants, and function $G(\cdot)$ is defined by formula:

$$G(\xi) := \frac{\exp(-|\xi|/\omega)}{2\omega}.$$

The constants C_+ and C_- have to be chosen so the jump conditions (40) are satisfied. They are found to be equal

$$C_+ = \int_{-\infty}^0 \{P_{\text{dr}}^c(S(\xi)) - P_{\text{im}}^c(S(\xi))\} G(\xi) d\xi,$$

$$C_- = \int_0^{+\infty} \{P_{\text{dr}}^c(S(\xi)) - P_{\text{im}}^c(S(\xi))\} G(\xi) d\xi.$$

Once the pressure is determined, S can be obtained from Equation (38). With P_{dr}^c and P_{im}^c given as in equalities (18), we find a linear equation for S :

$$L\dot{S}(x, t) = -bS(x, t) + b \int_{-\infty}^{+\infty} S(\xi, t) G(x - \xi) d\xi + \sigma \exp\left(-\frac{|x|}{\omega}\right) \text{sign}(x). \quad (41)$$

Fourier transformation over the spatial variable of this equation gives the following ordinary differential equation with respect to Fourier transform $S^F(k, t)$ of $S(x, t)$:

$$L\dot{S}^F(k, t) = -bS^F(k, t) \frac{\omega^2 k^2}{1 + \omega^2 k^2} + \frac{2\sigma \omega^2 k}{1 + \omega^2 k^2} i \quad (42)$$

with initial condition

$$S^F(k)|_{t=0} = 2\pi \frac{S_+ + S_-}{2} \delta(k) + \frac{S_+ - S_-}{2} \frac{2i}{k},$$

where $i = \sqrt{-1}$.

Equation (42) can be easily solved, and after inverse Fourier transformation, we obtain (19).

In order to prove that formula (19) is the solution of the problem under consideration, we must determine the sign of \dot{S} and verify the assumption on location of drainage and imbibition zones. Let us first calculate \dot{S} at $t = 0$. To this end, equation (41) is more suitable than expression (19). As a result of direct calculations, we get

$$\dot{S}|_{t=0} = -\frac{b}{L} \left(\frac{S_+ - S_-}{2} - \frac{\sigma}{b} \right) \exp\left(-\frac{|x|}{\omega}\right) \text{sign}(x).$$

One can see that, at least at initial instance, the assumption on location of drainage and imbibition zones holds if

$$\frac{S_+ - S_-}{2} - \frac{\sigma}{b} > 0.$$

This inequality is equivalent to the relation $P_{dr}^c(S_+) < P_{im}^c(S_-)$ which was indeed the underlying assumption. As explained earlier, in the opposite case, the solution is trivial and saturation field (17) is a steady state of the system as it was in Philip's problem.

Next, we differentiate Equation (41) with t in order to get an equation for \dot{S} . We only need to consider $x > 0$ because \dot{S} is an odd function of x due to (19). On the set of odd functions, the equation for \dot{S} can be reduced to the following form

$$L \frac{d}{dt} \dot{S}(x, t) = -b \dot{S}(x, t) + b \int_0^{+\infty} \dot{S}(\xi, t) [G(x - \xi) - G(x + \xi)] d\xi. \quad (43)$$

We consider this equation in the interval $0 \leq x < +\infty$, posing \dot{S} equal to its right limit at $x = 0$. For $x \geq 0$ and $\xi \geq 0$, the term $[G(x - \xi) - G(x + \xi)]$ is non-negative. Then for any $x \geq 0$ and any t , the equation (43) is followed by the inequality

$$L \frac{d}{dt} \dot{S}(x, t) \leq -b \dot{S}(x, t) + b \max\{\dot{S}\} \cdot \int_0^{+\infty} [G(x - \xi) - G(x + \xi)] d\xi,$$

where the maximum of \dot{S} is taken over all $x \geq 0$. The integral in the last term of this inequality can be easily calculated. It is equal to $1 - \exp\{-x/\omega\}$. If the maximum takes place at $x = x_{\max}$ then

$$L \frac{d}{dt} \dot{S}(x_{\max}, t) \leq -b \dot{S}(x_{\max}, t) \exp\{-x_{\max}/\omega\}.$$

This inequality provides a sort of maximum principle. It means that the function \dot{S} can not become positive if it was negative initially. Since we have already got the desired sign of the rate of saturation at initial instance, the drainage regime for all $x > 0$ will take place afterwards. The imbibition regime for all negative x could be established in the same way.

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