1.03.1 Introduction

Earth’s normal modes, or free oscillations, are standing waves along the surface and radius of the Earth. These standing waves only exist at discrete frequencies and are similar to the different ‘tones’ of musical instruments. The study of the Earth’s free oscillations is fundamental to seismology, as it is a key part of the theory of the Earth’s dynamic response to external or internal forces. Essentially, the same theory is applicable to phenomena as diverse as post-seismic relaxation, analysis of seismic surface waves and body waves. The study of free oscillations per se is concerned with analyzing and extracting information at very long periods (\(\sim 3000–100 \text{ s}\) period) since in this range of periods the intrinsic standing wave modes of oscillation are evident in seismic spectra. Normal mode peaks can be observed in the frequency spectrum of large earthquakes with magnitudes of, say, \(M_w > 7.5\), though the best observations are made for large and deep earthquakes such as the \(M_w = 8.5\) Bolivia earthquake of 1994. Such spectra contain important information about the large-scale structure of the Earth. For example, the strongest evidence that the inner core is solid (Dziewonski and Gilbert, 1971) and anisotropic (Romanowicz and Breger, 2000; Tromp, 1993; Woodhouse et al., 1986) comes from the study of free oscillations.

Free oscillations provide essential constraints on both the spherically symmetric ‘average’ Earth, and also on lateral variations in Earth structure due to heterogeneity in temperature, composition, and anisotropy. Modal data are particularly valuable in this regard because, unlike other kinds of seismic data, modal observables depend upon broad averages of the Earth’s
structural parameters, and are not nearly so affected by limitations of data coverage due to the uneven distribution of seismic events and stations. There is an enormous wealth of information yet to be extracted from long-period spectra; one has only to examine almost any portion of a seismic spectrum in detail to realize that current models often do not come close to providing adequate predictions. It is only from such very long-period data that it may be possible to obtain direct information on the three-dimensional distribution of density (Ishii and Tromp, 1999; Romanowicz, 2001; Trampert et al., 2004). Even very large scale, information on lateral variations in density has the potential to bring unique information to the study of convection and thermal and compositional evolution.

Very long-period spectra are also an essential element in the study of earthquakes, as it is only by using data at the longest periods that it is possible to determine the overall moment of very large events. For example, estimates of the moment of the great Sumatra earthquake based on mantle waves, even at periods of several hundred seconds, significantly underestimate the true moment, as the length and duration of the rupture make it possible to gauge the true, integrated moment only by using data at the longest seismic periods (e.g., Park et al., 2005).

Figure 1(a) shows an example of data and theoretical amplitude spectra computed for the spherically symmetric nonrotating PREM model (Dziewonski and Anderson, 1981). Modes appear as distinct peaks in the frequency domain. For higher frequencies, the modes are more closely spaced and begin to overlap. The theoretical peaks appear at frequencies very close to the observed peaks. However, the observed peaks are distorted in shape and amplitude due to three-dimensional effects. For example, mode $1S_4$ is split into two peaks in the data spectrum, which is not seen in the theoretical spectrum. When adding the effects of Earth rotation and ellipticity of figure (Figure 1(b)), the peaks in the synthetic spectrum start looking more similar to the data; mode $1S_4$ has become broader in the synthetic spectrum. Even better agreement is obtained for a fully three-dimensional Earth model (Figure 1(c)), for example, the pair of modes $3S_2$–$2S_3$ now looks very similar in the data and synthetic spectra. However, there are also still significant differences demonstrating that a lot can still be learnt from studying long-period normal mode spectra. Not shown here, but equally important in studying both Earth structure and earthquakes, is the phase spectrum. Examples illustrating this are shown in later sections.

Normal mode studies represent the quest to reveal and to understand the Earth’s intrinsic vibrational spectrum. However, this is a difficult quest, because it is only at the very longest periods ($\geq 100$ s, say) that there is the possibility of obtaining data of sufficient duration to make it possible to achieve the necessary spectral resolution. Essentially the modes attenuate before the many cycles necessary to establish a standing wave pattern have elapsed. Thus, in many observational studies, over a wide range of frequencies, the normal mode representation has the role, primarily, of providing a method for the calculation of theoretical seismograms. Although observed spectra contain spectral peaks, the peaks are broadened by the effects of attenuation in a path-dependent way. Thus, rather than making direct measurements on observed spectra, the analysis needs to be based on comparisons between data and synthetic spectra, in order to derive models able to give improved agreement between data and synthetics.

The use of normal mode theory as a method of synthesis extends well beyond the realm normally thought of as normal mode studies. For example, it has become commonplace to calculate global body wave theoretical seismograms by mode summation in a spherical model, to frequencies higher than 100 mHz (10 s period). Typically such calculations can be
done in seconds on an ordinary workstation; the time, of course, depending strongly upon the upper limit in frequency and on the number of samples in the time series. The advantage of the method is that all seismic phases are automatically included, with realistic time and amplitude relationships. Although the technique is limited (probably for the foreseeable future) to spherically symmetric models, the comparison of such synthetics with data provides a valuable tool for understanding the nature and potential of the observations and for making measurements such as differences in timing between data and synthetics, for use in tomography. Thus, the period range of applications of the normal mode representation extends from several thousand seconds to ~5 s. In between these ends of the spectrum is an enormous range of applications: studies of modes per se, surface wave studies, analysis of overtones and long-period body waves, each having relevance to areas such as source parameter estimation and tomography. Figure 2 shows an example of synthetic and data traces to illustrate this. The normal mode synthetic is calculated by summing all modes with period ≥6 s. The body waves P, PP, S, and SS are visible in the early part of the synthetic and data seismograms, and the surface waves arrive in the later part. It is important to note that S diffracted waves can also be obtained using normal mode summation, as normal modes form a complete basis set and thus all waves existing in a spherical Earth model are automatically included when all modes are summed up to a certain period. Differences between the synthetic and observed data seismograms can be attributed to unmodeled three-dimensional heterogeneity.

There are a number of excellent sources of information on normal modes theory and applications. The comprehensive monograph by Dahlen and Tromp (1998) provides in-depth coverage of the material and an extensive bibliography. An earlier monograph by Lapwood and Usami (1981) contains much interesting and useful information, treated from a fundamental point of view, as well as historical material about early theoretical work and early observations. A review by Takeuchi and Saito (1972) is a good source for the ordinary differential equations for spherical Earth models and methods of solution. Other reviews are by Gilbert (1980), Dziewonski and Woodhouse (1983b), and Woodhouse (1996). A review of normal mode observations can be found in Chapter 1.04. Because of this extensive literature, we tend in this article to expand on some topics that have not found their way into earlier reviews but are nevertheless of fundamental interest and utility. We will first derive the equations of motion using Hamilton’s principle, and introduce the concept of generalized spherical harmonics. We will then show how to obtain the

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**Figure 2** Data and normal mode synthetic for a vertical component record at station HRV following an event in Hawaii on 15 October 2006. The epicentral distance is 73°. For the calculation of the synthetic seismogram, the Harvard/Lamont quick-CMT centroid location and moment tensor parameters were used (G. Ekström, www.globalcmt.org).
solution for a spherically symmetric Earth and the numerical method used for calculating the modes. Finally, we will explain the calculation of synthetic seismograms in spherical and three-dimensional Earth models and discuss the characteristics of the normal mode spectrum.

1.03.2 Equations of Motion and Hamilton’s Principle

To a good approximation, except in the vicinity of an earthquake or explosion, seismic displacements are governed by the equations of elasticity. At long periods self-gravitation also plays an important role. The equations of motion can be derived using conservation laws (e.g., Woodhouse, 1996). Alternatively, they can be derived from a variational principle, which significantly lessens the algebra involved. First, the Lagrangian, which is the total kinetic energy minus the total potential energy, is defined. Then the equations of motion and boundary conditions are obtained by requiring that the Lagrangian be stationary with respect to small variations in displacement and gravitation. There are two alternative ways to use a variational principle, depending on the domain in which the displacement and gravitation are varied. In the frequency domain, Rayleigh’s principle has been used (Woodhouse and Dahlen, 1978), and in the time domain Hamilton’s principle can be used. Here, we show how the equations of motion arise from Hamilton’s principle by first defining the Lagrangian and then varying the Lagrangian with respect to displacement and gravity to get the equations of motion and corresponding boundary conditions.

1.03.2.1 Definition of the Earth Model and Lagrangian

Consider a material which is initially in equilibrium under self-gravitation. Each particle of the material is labeled by Cartesian coordinates \( x \) (\( i = 1, 2, 3 \)), representing its initial position. The material undergoes time-dependent deformation in which the particle initially at \( x \) moves to \( r \equiv r(t, x) \) where \( t \) is time (Figure 3). A hyperelastic material is defined as one in which there exists an internal energy density function which is a function of the Green strain tensor

\[
\varepsilon_{ij} = \frac{1}{2} \left( r_{,ij} r_{,ij} - \delta_{ij} \right)
\]  

where \( r_{,ij} \equiv \partial r_{ij}/\partial x_{ij} \), \( \delta_{ij} \) is the Kronecker-delta; summation over repeated indices is assumed. Thus, we introduce the internal elastic energy function, per unit mass, \( E(x, e, s) \), where \( s \) is specific entropy. We shall be concerned only with isentropic deformations, and can henceforth omit the dependence on \( s \). Note that \( E \) is represented as a function of the coordinates \( x \), which label a specific material particle. \( E(x, e) \) is regarded as a given function characterizing the elastic properties of the material. This form of the internal energy function, which forms the basis of finite theories of elasticity, as well as the theory of linear elasticity which we need here, arises from the very general consideration that elastic internal energy should not change as a result of rigid rotations of the material.

The gravitational field is characterized by a potential field \( \phi(r) \), which satisfies Poisson’s equation

\[
\frac{\partial^2 \phi}{\partial r^2} = -4\pi G \rho(r)
\]  

where \( \rho \) is the density. We state this equation in terms of the coordinates \( r_\nu \) as it represents an equation valid in the current configuration of the material.

The Lagrangian governing motion of the elastic-gravitational system (total kinetic energy minus total potential energy) is

\[
L = \int \left( \frac{1}{2} \rho \dot{r}^2 - \rho E(x, e) - \frac{1}{2} \rho \phi \right) d^3r dt
\]  

where \( \dot{\cdot} \) represents the material time derivative (i.e., the derivative with respect to \( t \) at constant \( x \)). The three terms of the integrand represent the kinetic energy, (minus) the elastic energy, and (minus) the gravitational energy, respectively, making use of the fact that the gravitational energy released by assembling a body from material dispersed at infinity is

\[
-\int \frac{1}{2} \rho \phi d^3r.
\]

The integral in eqn [3] is over the volume occupied by the Earth, which is taken to consist of a number of sub-regions with internal interfaces and an external free surface. Hamilton’s principle requires that the Lagrangian \( L \) be stationary with respect to variations in displacement \( \delta r \), subject to the constraint that \( \phi \) is determined by eqn [2], and also to the requirement of mass conservation:

\[
\rho d^3r = \rho^0 d^3x
\]  

where \( \rho^0 = \rho^0(x) \) is the initial density. That is

\[
\rho = \rho^0 / J
\]  

where \( J \) is the Jacobian

\[
J = \frac{\partial(r_1, r_2, r_3)}{\partial(x_1, x_2, x_3)}
\]  

The gravitational constraint [2] can be incorporated into the variational principle by introducing a field \( \eta \) which acts as a Lagrange multiplier (e.g., Seliger and Whitham, 1968):

\[
L' = \int \left\{ \frac{1}{2} \rho \dot{r}^2 - \rho E(x, e) - \frac{1}{2} \rho \phi + \eta \left( \frac{\partial^2 \phi}{\partial r^2} - 4\pi G \rho \right) \right\} d^3r dt
\]  

the term involving \( \eta \) vanishes by eqn [2], and therefore, \( L' \) is stationary with respect to variations in \( \eta \). If we also require that \( L' \) be stationary under variations in gravitational field \( \delta \phi \) we obtain the Euler–Lagrange equation for \( \eta \):

\[
\frac{\partial^2 \eta}{\partial r^2} = \frac{1}{2} \rho
\]  

which can be satisfied by setting \( \eta = \phi/8\pi G \). Thus, we obtain
L' = \int \left\{ \frac{1}{2} \rho \dot{r}_i \dot{r}_i - \rho_0 E(x, \mathbf{c}) - \rho_0 \phi - \frac{1}{8\pi G} \frac{\partial \phi}{\partial r_j} \frac{\partial \phi}{\partial r_j} \right\} d^3r dt \quad [9]

Changing the spatial integration variables, making use of eqn [4], we may also write:

L' = \int \left\{ \frac{1}{2} \rho \dot{r}_i \dot{r}_i - \rho_0 E(x, \mathbf{c}) - \rho_0 \phi - \frac{1}{8\pi G} \frac{\partial \phi}{\partial r_j} \frac{\partial \phi}{\partial r_j} \right\} d^3x dt \quad [10]

where \( \phi' \) represents the gravitational potential at the fixed coordinate point \( x \). Notice that the first three terms of the integrand have been transformed by regarding \( r \) to be a function of \( x \) (at each fixed \( t \)) through the function \( r(x, t) \) which defines the deformation. However, the fourth term (which could be treated in the same way) has been transformed by renaming the dummy integration variables \( r_j \) to \( x_i \). Hence the need to introduce \( \phi' \) since \( \phi \) represents \( \phi(r(x,t)) \), which is different from \( \phi(x,t) \). The requirement that \( L' \) be stationary with respect to variations \( \delta r(x,t), \delta \phi(r,t) \) provides a very succinct, complete statement of the elasto-gravitational dynamical equations. In other words, \( L' \) will be stationary under variations in \( \delta r(x,t), \delta \phi(r,t) \) if and only if \( r, \phi \) are eigendifunctions of the associated Earth model.

### 1.03.2.2 Equations of Motion and Boundary Conditions

To obtain the partial differential equations for infinitesimal deformations, we approximate \( L' \) in the case that \( t_k = x_k + u_k(x,t) \) (Figure 3) and \( \phi' = \phi_0 + \epsilon \phi_1(x,t) \), where \( \epsilon \) is a small parameter. We seek to express the Lagrangian \( L' \) in terms of the fields \( u_k, \phi \), to second order in \( \epsilon \). We have, to second order in \( \epsilon \):

\[
\phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdot \cdot \cdot
\]

We expand \( \rho_0 E(x, \mathbf{e}) \) to second order in strain:

\[
\rho_0 E(x, \mathbf{e}) = a + \epsilon \epsilon_0 + \frac{1}{2} \epsilon_ijkl \epsilon_{ijkl}
\]

As a result of their definitions, as first and second derivatives of \( \rho_0 E(x, \mathbf{e}) \) with respect to strain, at zero strain, \( \epsilon_0 \) and \( \epsilon_ijkl \) possess the symmetries:

\[
\epsilon_ijkl = \epsilon_ijkl
\]

We use the notation \( \epsilon_0 \) since these expansion coefficients represent the initial stress field. The strain tensor \([11]\) is

\[
e_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} + \frac{1}{2} \epsilon^2 u_{i,k} u_{k,j} \right)
\]

Thus, the second order expansion of \( L' \) becomes:

\[
L' = \int \left\{ \frac{1}{2} \rho \dot{r}_i \dot{r}_i - \rho_0 \phi_0 - \frac{1}{8\pi G} \phi_0 \phi_0 \right\} d^3x dt
\]

where we have introduced

\[
A_{ijkl} = \delta_{i[j} \epsilon_{k]l} + \epsilon_{ijkl}
\]

and where the \( A_{ijkl} \) term in eqn [16] is the incremental Piola–Kirchhoff stress tensor. The Euler–Lagrange equations which require that \( L' \) be stationary with respect to variations \( \delta u_i, \delta \phi \), are given by

\[
\frac{\partial L'}{\partial u_i} = \left( \frac{\partial L'}{\partial u}_{,j} \right)_j \quad \text{and} \quad \frac{\partial L'}{\partial \phi} = \left( \frac{\partial L'}{\partial \phi}_{,i} \right)_i
\]

They must hold for each power of \( \epsilon \). Thus, the first order terms in eqn [16] give the two following equations, which represent the terms in \( L' \) that are independent of \( \epsilon \) and do not contribute to the variation, and thus can be omitted. When eqns [19] and [20] are satisfied, there remain only the second order terms:

\[
L'' = \int \left\{ \frac{1}{2} \rho \dot{u}_i \dot{u}_i - A_{ijkl} u_{i,j} u_{k,l} - \rho_0 u_i \phi_0^{ij} - 2 \rho_0 u_i \phi_0^{al} \right\} d^3x dt
\]

In eqn [21] we omit the factor \( \epsilon^2 \), absorbing the small parameter into the definitions of the fields \( u, \phi \). The Euler–Lagrange equations for \( L'' \) (similar to eqn [18]) corresponding to variations \( \delta u_i, \delta \phi \) give the equations of motion:

\[
\rho \left( \dot{u}_i + \phi_0^{ij} \partial_j \mu \right) = \left( A_{ijkl} \mu_{k,l} \right)_j
\]

The same equations can alternatively be derived using conservation laws, see for example Woodhouse (1996).

The variational principle also leads to certain natural boundary conditions at the free surface (i.e., the ocean surface or the outer surface of the solid Earth in the absence of an ocean) and at internal boundaries in the case that \( r(x,t) \) is required to be continuous at such boundaries. These internal boundaries are so-called welded boundaries, such as the Mohorovicic discontinuity between the crust and mantle and the 660 km discontinuity between the upper and lower mantle. These are as given below. We also wish to include the case that the model contains fluid regions, having free-slip, boundary conditions at their interfaces with solid regions – the so-called frictionless boundaries, which exist at the ocean floor, the inner core boundary, and the outer core boundary. The correct treatment of such boundaries introduces complications that, in the interests of giving a concise account, we do not analyze in detail here. Woodhouse and Dahlen (1978) show that it is necessary to include additional terms in the Lagrangian to account for the additional degrees of freedom corresponding to slip (i.e., discontinuous \( u_i \)) at such boundaries. The stress boundary conditions are most conveniently stated in terms of the vector \( t_i \) defined on the boundary

\[
t_i = A_{ijkl} u_{k,l} - n_i (\sigma^0 u_k)_{,k} + \epsilon_0 u_i n_k
\]

where \( n_i \) is the unit normal to the boundary, and where the semicolon notation, for example, \( u_{;w} \) is used to indicate differentiation in the surface: \( u_{;w} = u_{,w} - n_p u_{,p} \), \( n_k \). We also require the gravitational potential to vanish at infinity. The complete set of boundary conditions is
Welded: \[
\begin{bmatrix}
    i_n^0 & = 0, \\
    -i_{n^0}^+ & = 0, & [t_i]^+ = 0,
\end{bmatrix}
\]

Frictionless: \[
\begin{bmatrix}
    i_n^0 & = 0, \\
    i_n^0 & = m_n^0, & [u_k^0]^+ = 0, & [t_i]^+ = 0, & t_i = n_t k n_t.
\end{bmatrix}
\]

Free: \[
\begin{bmatrix}
    i_n^0 & = 0, & t_i = 0.
\end{bmatrix}
\]

All: \[
\begin{bmatrix}
    [\phi^0]^+ & = 0, \\
    [\phi^1]^+ & = 0, & [\phi^1] & = 0, \\
    [\phi^1 + 4\pi C_0^0 u]_{ji} & = 0,
\end{bmatrix}
\]

Infinity: \[
\begin{bmatrix}
    \phi^0 & \to 0, \\
    \phi^1 & \to 0,
\end{bmatrix}
\]

where \([\cdot]^+\) represents the discontinuity of the enclosed quantity across the boundary. Equations [22]–[25] govern the unknown displacement field \(u_i(x, t)\) and gravity field \(\phi(x, t)\) which represent the possible free oscillations of the Earth. All other quantities \(\rho^0, \Gamma_i\), \(A_{ijkl}\) are regarded as given parameters of the Earth model subject to the equilibrium conditions [19] and [20].

1.03.2.3 The Seismic Source

In order to excite the normal modes, we need to apply a force, or seismic source. In the presence of an applied force distribution \(f_i = f_i(x, t)\), per unit volume, the equation of motion [22] becomes

\[
\rho^0 \left( \ddot{u}_i + \phi^0_{ij} \phi^0_{ij} u_j \right) = (A_{ijkl} u_k)_i = f_i
\]

[26]

Taking the Fourier transform in time, we shall also write

\[
\rho^0 \left( -\omega^2 u_i + \phi^0_{ij} \phi^0_{ij} u_j \right) - (A_{ijkl} u_k)_i = f_i
\]

[27]

where \(\omega\) is the frequency. We shall employ the Fourier transform pair:

\[
u_i(x, \omega) = \int_{-\infty}^{\infty} u_i(x, t) e^{-i\omega t} dt, \quad u_i(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \nu_i(x, \omega) e^{i\omega t} dt
\]

[28]

Here, and in subsequent equations, we rely on the context to distinguish between time-domain and frequency-domain quantities, adopting the convention that if \(\omega\) appears in an equation then all functions appearing are the Fourier transforms of the original, time-dependent functions.

Earthquake sources can be modeled by choosing particular force distributions \(f_i\). The problem of determining the equivalent body force distribution \(f_i\), to represent (prescribed) slip on an earthquake fault was originally solved by Burridge and Knopoff (1964) using the elastodynamic representation theorem. A very general and, at the same time, simple approach to the problem of determining body force equivalents is that of Backus and Mulcahy (1976). They argue that an earthquake occurs as a result of the failure of the assumed constitutive law, in the linear case Hooke’s law, relating stress and strain. This leads them to introduce a symmetric tensor quantity \(\Gamma_{ij} = \Gamma_p(x, t)\), called the stress gluit, which represents the failure of Hooke’s law to be satisfied. Importantly, \(\Gamma_{ij}\) (i) will be zero outside the fault zone, (ii) will be zero at times before the earthquake, and (iii) will have vanishing time derivative at times after slip has ceased. Thus, the stress rate, \(\dot{\Gamma}_{ij}\), is compact in space and time. The earlier concept of stress-free strain, due to Eshelby (1957), is a closely related one. The strain, for slip on a fault, contains \(\delta\)-function terms at the fault, because the displacement is discontinuous. The stress, on the other hand, is finite on the fault. Thus, there exists a stress glut – a failure of the stress to satisfy Hooke’s law, and a stress-free strain, that is, a component of the strain field that is not reflected in the stress. The existence of a nonvanishing stress glut, \(\Gamma_{ij}\), leads us to replace eqn [22] by

\[
\rho^0 \left( \ddot{u}_i + \phi^0_{ij} \phi^0_{ij} u_j \right) = (A_{ijkl} u_k)_i = f_i
\]

[29]

and thus comparing [26] with [30], the equivalent body force distribution is found to be \(f_i = -\Gamma_{ij}\). Because \(\Gamma_{ij}(x, t)\) is compact in space and time, it is appropriate for calculations at long period and long wavelength to replace it by a \(\delta\)-function in space and time. Defining the moment tensor

\[
M_{ij} = \int_V \Gamma_{ij}(x, \omega) d^3x = \int_\infty \int_\infty \Gamma_{ij}(x, t) d^3x dt
\]

[31]

where \(V\) is the source volume – the region over which \(\Gamma_{ij}\) is nonzero – a suitable form for \(\Gamma_{ij}\) is \(\Gamma(x, t) = M_{ij} \dot{\delta}^j(x - x_s) H(t - t_s)\), where \(H(t)\) is the Heaviside step function, and where \(x_s, t_s\) are the source coordinates. Thus \(f_i = -M_{ij} \partial^j (x - x_s) H(t - t_s)\). In what follows we shall consider the more general point source

\[
f_i = (F_i - M_{ij} \dot{\delta}^j(x - x_s)) H(t - t_s)
\]

[32]

in which \(M_{ij}\) is not necessarily symmetric, recognizing that for sources not involving the action of forces external to the Earth, so-called indigenous sources such as earthquakes, must be symmetric and \(M_{ij}\) must be zero. The solution for a point force \(F_i\) is of fundamental theoretical interest since the solution in this case is the Green’s function for the problem, which can be used to construct solutions for any force distribution \(f_i\). Non symmetric \(M_{ij}\) corresponds to a source which exerts a net torque or couple on the Earth, such as meteorite impacts.

1.03.2.4 Viscoelasticity and Intrinsic Attenuation

The hyperelastic constitutive law based on the internal energy function \(E(x, e)\) needs to be modified to include the effects of energy loss due to such effects as grain boundary sliding and creep. Such anelastic effects lead to dissipation of energy (i.e., conversion of elastic stored energy into heat) and thus to the decay, or attenuation of seismic waves. In addition, they are responsible for such effects as post-seismic relaxation, and the theory developed here is in large part applicable to this problem also. A generalization of the constitutive law which retains linearity is the viscoelastic law, which supposes that stress
depends not only on the strain at a given instant, but also on the strain history. This can be written as

\[ t_i(t) = \int_{-\infty}^{\infty} c_{ijkl}(t - t')u_{k_l}(t') \, dt' \]  \[ 33 \]

where \( t_i \) is incremental stress. (In fact it can be shown that the true increment in stress, at a material particle, includes terms in the initial stress: \( t_i = c_{ijkl}u_{k_l} + \frac{\partial t}{\partial t} = c_{ijkl}u_{k_l} - t_i^0u_{k_l} \), but this makes no difference to the discussion here.) Thus, the elastic constants become functions of time, relative to a given time \( t \) at which the stress is evaluated. Importantly, since stress can depend only upon past times, \( c_{ijkl}(t) \) must vanish for negative values of \( t \); that is, it must be a causal function of time. In order to recover the strict Hooke’s law, \( t_i = c_{ijkl}u_{k_l} \) we need \( c_{ijkl}(t) = c_{ijkl}(0) \) (we are distinguishing here between \( c_{ijkl} \) unadorned, which has the units of stress, and \( c_{ijkl}(t) \), which has the units of stress/time). In the frequency domain, using the convolution theorem,

\[ t_i(\omega) = c_{ijkl}(\omega)u_{k_l}(\omega) \]  \[ 34 \]

Because \( c_{ijkl}(t) \) is a causal function, its Fourier transform

\[ c_{ijkl}(\omega) = \int_{-\infty}^{\infty} c_{ijkl}(t)e^{-i\omega t} \, dt \]  \[ 35 \]

will be analytic, that is, will have no singularities, in the lower half of the complex \( \omega \)-plane, as the integral \[35\] will converge unconditionally in the case that \( \omega \) possesses a negative imaginary part. From eqn \[35\] \( c_{ijkl}(\omega) = c_{ijkl}(-\omega^*) \). For our purposes here, the key conclusion is that in the frequency domain, it makes virtually no difference to the theory whether the material is hyperelastic, or viscoelastic, as we have simply everywhere to substitute \( c_{ijkl}(\omega) \) for \( c_{ijkl} \). In fact there is even no need to introduce a new notation, but only to remember that now \( c_{ijkl} \) can represent a complex quantity depending on \( \omega \) and analytic in the lower half of the complex \( \omega \)-plane. When writing equations in the time domain, we have to remember that now \( c_{ijkl} \) can be a convolution operator, acting on the strain. (It may be remarked that the above derivation of the equations of motion, based on Hamilton’s principle, is in need of modification if \( E(x, e) \) does not exist. We do not quite know how to do this, but a monograph by Biot (1965) discusses the use of variational principles in the presence of anelastic effects.) Kanamori and Anderson (1977), and references cited therein, is a good source for further information on this topic.

### 1.03.2.5 Final Equation of Motion

It is often useful to summarize the equations and the boundary conditions by a single simple equation:

\[ (\mathcal{H} + \rho^0\partial_t^2)u = f \]  \[ 36 \]

where \( \mathcal{H} \) represents the integro-differential operator corresponding to the left side of eqn \[26\], omitting the term in \( \rho^0u \). Here, \( \phi^1 \) is thought of as a functional of \( u \), that is, as the solution of Poisson’s equation \[23\], corresponding to a given \( u \) \((x, t)\), together with the boundary conditions relating to \( \phi^1 \) in eqn \[25\]. Thus, \( \mathcal{H}u \) incorporates the solution of eqn \[23\]. In the attenuating case, \( \mathcal{H} \) also includes the time-domain convolutions arising from the viscoelastic rheology.

### 1.03.3 Spherical Harmonics and Generalized Spherical Harmonics

Free oscillations involve the motion of the whole Earth, so we need to solve the equations of motion in spherical coordinates. The reduction of these equations in spherical coordinates is most easily accomplished through the use of the generalized spherical harmonic formalism (Phinney and Burridge, 1973). These enable any tensor field to be readily expanded in spherical harmonics, facilitating the derivation of all modal equations in spherical harmonics. Here, we describe how this formalism is used, giving some key results without derivation. We shall use a standard set of Cartesian coordinates \((x, y, z)\) and spherical coordinates \((r, \theta, \phi)\) related by

\[
\begin{align*}
x_1 &= x = r \sin \theta \cos \phi, \\
x_2 &= y = r \sin \theta \sin \phi, \\
x_3 &= z = r \cos \theta
\end{align*}
\]  \[ 37 \]

Unit vectors in the coordinate directions are given by

\[
\begin{align*}
\mathbf{i} &= [\cos \theta \cos \phi, \sin \theta \cos \phi, \cos \theta], \\
\mathbf{j} &= [\cos \theta \sin \phi, \sin \theta \sin \phi, -\sin \theta], \\
\mathbf{k} &= [-\sin \phi, \cos \phi, 0]
\end{align*}
\]  \[ 38 \]

Spherical components of vectors and tensors will be written, for example, \( u_\theta = u_\theta \mathbf{\hat{e}}_\theta = \mathbf{t}_\theta \mathbf{\hat{e}}_\theta \).

#### 1.03.3.1 Spherical Harmonics

We define the complex spherical harmonics following Edmonds (1960)

\[
Y_l^m(\theta, \phi) = (-1)^m \left[ \frac{(l - m)!}{(l + m)!} \right]^{1/2} P_l^m(\cos \theta) e^{im\phi}
\]  \[ 39 \]

where angular order \( l = 0, 1, 2, \cdots \), azimuthal order \( m = -l, -l + 1, \cdots, l \) and \( P_l^m(x) \) are the associated Legendre functions

\[
P_l^m(x) = \frac{(1 - x^2)^{m/2}}{2^l l!} \frac{d^l}{dx^l} \left( (x^2 - 1) \right)
\]  \[ 40 \]

The spherical harmonics \( Y_l^m(\theta, \phi) \) satisfy the orthogonality relation:

\[
\int_{-\pi}^{\pi} \int_{0}^{\pi} Y_l^m(\theta, \phi)^* Y_l'^m(\theta, \phi) \sin \theta d\theta d\phi = \delta_{l'l} \delta_{m'm}
\]  \[ 41 \]

Figure 4 shows examples of the surface expression of the first few spherical harmonics \( Y_l^m(\theta, \phi) \). The number of nodal lines in the \( \theta \)-direction is given by \( l - m \), so for \( m = 0 \) there will be \( l \) nodal lines in the \( \theta \)-direction. The number of nodal lines in the \( \phi \)-direction is given by \( 2m \), that is, for \( m = 0 \) there are no nodal lines in the \( \phi \)-direction.

#### 1.03.3.2 Generalized Spherical Harmonics

The prescription provided by the generalized spherical harmonic formalism is first to define the spherical contravariant components of the vectors and tensors that appear, and then to expand their dependence on \((\theta, \phi)\) in terms of complete sets of functions appropriate to the particular component. For a tensor of rank \( p \), having spherical components \( s_{l_1 l_2 \cdots l_p} \), spherical contravariant components are defined by
The spherical harmonics $Y_l^m$, showing real and imaginary parts for $m \geq l$, for angular order $l = 1 - 4$. Dashed lines indicate the zero value contour, that is, the ‘nodal lines’ of the spherical harmonic. Blue is positive and red is negative.

The real quantities $d_{Nm}^{l}(\theta) = P_{Nm}(\cos \theta)$ are rotation matrix elements employed in the quantum mechanical theory of angular momentum (Edmonds, 1960); thus $Y_l^m$ vanish for $N$ or $m$ outside the range $-l$ to $l$. The spherical harmonic degree $l$ characterizes a group representation of the rotation group; as a consequence, tensor fields that are spherically symmetric have only the term with $l = 0$, $N = 0$, $m = 0$. The property of the rotation matrix elements

$$d_{Nm}^{l}(\theta) = \begin{cases} 1 & \text{if } m = N \text{ and } l \geq |N| \\ 0 & \text{otherwise} \end{cases}$$

is a very useful one, for example, for calculations of source excitation coefficients, when it is required to evaluate spherical harmonic expressions for $\theta = 0$ (see the succeeding text). Equation [46] says that regarded as $(2l+1) \times (2l+1)$ matrix, having row index $N$ and column index $m$, $d_{Nm}^{l}(\theta)$ is the unit matrix. Matrices $d_{Nm}^{l}(\theta)$ have symmetries

$$d_{Nm}^{l}(\theta) = d_{Nm}^{l}(\theta) = (-1)^{m-N} d_{Nm}^{l}(\theta),$$

from which follows the relation $Y_{l}^{Nm}(\theta, \phi)^* = (-1)^{m-N} Y_{l}^{-N-m}(\theta, \phi)$, where * denotes the complex conjugate. $Y_l^m$ satisfy the orthogonality relation

$$\int \int Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) \, \sin \theta \, d\theta \, d\phi = \frac{4\pi}{2l+1} \delta_{ll'} \delta_{mm'}$$

where $\delta_{ll'}$ and $\delta_{mm'}$ are Kronecker delta functions.
The major advantage of this formalism is that, in a spherically symmetric system, it enables vector and tensor relations to be transformed into relations for spherical harmonic coefficients, by the application of a straightforward set of rules. Importantly, the resulting relations (i) are true for each value of $l$ and $m$ separately and (ii) are the same for each spherical harmonic order $m$.

1.03.3 Wigner 3-j Symbols

The treatment of aspherical systems requires results for the products of spherical harmonic expansions. It can be shown that

$$Y_{l_1}^{N_1}m_1 Y_{l_2}^{N_2}m_2 = (-1)^{N_1 + N_2 - m_1 - m_2} \sum_{i=0}^{l_1+l_2} (2l+1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & i \end{pmatrix} Y_{l_3}^{N_3}m_3$$

where the so-called Wigner 3-j symbols are the (real) quantities arising in the theory of the coupling of angular momentum in quantum mechanics (see Edmonds, 1960). These satisfy

$$\frac{d}{dr} s_{lm}^{z_1 - z_2} = \delta_{z_1 - z_2}^{r} \frac{d}{dr} s_{lm}^{z_1 - z_2}$$

which leads to the following rule for the expansion coefficients of the gradient of a tensor:

$$s_{lm}^{z_1 - z_2} = \frac{d}{dr} s_{lm}^{z_1 - z_2}$$

The 3-j symbol is symmetric under even permutations of its columns, and either symmetric or antisymmetric under odd permutations, depending upon whether $l_1 + l_2 + l_3$ is even or odd. This has the consequence that if the sum of the $l$'s is odd and if the $m$'s are zero, the 3-j symbol is 0. Equation [52] leads to the following result for the spherical harmonic coefficients of the product of two tensor fields; suppose that $c_{i_{l_1} i_{l_2} i_{l_3}} = 2 \delta_{i_{l_1} - i_{l_2} - i_{l_3}}$, then

$$c_{i_{l_1} i_{l_2} i_{l_3}} = \sum_{i_{l_1} i_{l_2} i_{l_3}} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

The summations here are over all values of $l$, $l_1$, $m_1$, however, it is a finite sum by virtue of the fact that the terms vanish for values outside the ranges specified in eqn [53].
1.03.3.4 Application: Point Force Distribution

As an application of the spherical harmonic formalism, here we consider the expansion of the point force distribution [32] in spherical harmonics. It will be sufficient to locate the source at time \( t_0 = 0 \) and at a point on the positive \( z \)-axis, that is, at \( x = 0, y = 0, z = r_0 \), where \( r_0 \) is the source radius. Because \( \theta = 0 \) is a singular point in the spherical coordinate system, we shall consider the limit as the source approaches the ‘pole,’ \( \theta_i = 0 \), along the ‘meridian,’ \( \phi_i = 0 \). In the frequency domain, eqn \[32\] becomes

\[
 f_i = \frac{1}{i \omega} (F_i - \mathcal{M}_0 \delta \dot{r}_0) r^2 \csc \theta \delta(\theta - \theta_0) \delta(\phi) \delta(r - r_0) \quad \text{[55]}
\]

In the limiting process, \( \theta_i \to 0 \), we shall take \( F_i, \mathcal{M}_0 \) to have constant spherical components \( F_{00}, \mathcal{M}_{m0}, \mathcal{M}_{0m} \) etc. As \( \theta_i \to 0 \), the \( \theta, \phi \), and \( r \) directions end up pointing along the \( x, y, z \) directions, respectively, of the global Cartesian coordinate system (e.g., see eqn \[38\]), and thus although \( \theta, \phi \) are undefined at \( \theta = 0 \), we can nevertheless interpret the spherical components \( F_{00}, \mathcal{M}_{0m} \) etc. as representing the components \( F_x, \mathcal{M}_{x0} \) etc. of the point force and moment tensor in the global Cartesian system. Let \( \chi = (i \omega)^{-1} r^{-2} \csc \theta \delta(\theta - \theta_0) \delta(\phi) \delta(r - r_0) \), so that [55] can be written: \( f_i = \mathcal{F}_0 - \partial_t \mathcal{M}_{0z} \). The spherical contravariant components of \( F_i \) and \( \mathcal{M}_{0i} \) using a matrix representation of eqn [43], are given by

\[
 \begin{bmatrix}
 F_r \\
 F_\theta \\
 F_\phi \\
 \end{bmatrix} = \begin{bmatrix}
 \frac{1}{\sqrt{2}} & i \sqrt{2} & 0 \\
 0 & 0 & 1 \\
 -i \sqrt{2} & \sqrt{2} & 0 \\
 \end{bmatrix} \begin{bmatrix}
 F_0 \\
 F_\theta \\
 F_\phi \\
 \end{bmatrix}
\]

\[
 \begin{bmatrix}
 M_{-1} & M_{-0} & M_{-1} \\
 M_{0-} & M_{00} & M_{0+} \\
 M_{1-} & M_{10} & M_{1+} \\
 \end{bmatrix} = \begin{bmatrix}
 1 & i & 0 \\
 \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
 0 & 0 & 1 \\
 -i \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
 \end{bmatrix} \begin{bmatrix}
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 \end{bmatrix}
\]

\[
 \left( \frac{1}{\sqrt{2}} \begin{bmatrix}
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 \end{bmatrix} \right)_{m+0} \left( \frac{1}{\sqrt{2}} \begin{bmatrix}
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 \end{bmatrix} \right)_{m+0} \left( \frac{1}{\sqrt{2}} \begin{bmatrix}
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 M_{00} & M_{00} & M_{00} \\
 \end{bmatrix} \right)_{m+0} = \quad \text{[56]}
\]

The spherical harmonic expansion coefficients of \( F^r, \mathcal{M}^{r,-} \), for which we use the notation \( (F^r)_0^{lm}, (\mathcal{M}^{r,-})_0^{lm} \), are immediate using [48], integrating out the \( \delta \)-functions contained in \( \chi \). We obtain

\[
 (F^r)_0^{lm} = \frac{2l+1}{4\pi \omega} \int \frac{\delta(r-r_0)}{r^2} Y_r^{lm}(\theta_0, 0) F^r \\
 = \frac{2l+1}{4\pi \omega} \int \frac{\delta(r-r_0)}{r^2} d_0^{lm}(\theta_0) F^r \\
 = \frac{2l+1}{4\pi \omega} \int \frac{\delta(r-r_0)}{r^2} F^r \text{ if } m = z \text{ and } l \geq |z| \\
 0 \text{ otherwise} \quad \text{[57]}
\]

where \( \theta_i \) has been set to \( 0 \) in the second line, using [46]. Similarly,

\[
 (\mathcal{M}^{r,-})_0^{lm} = \frac{2l+1}{4\pi \omega} \int \frac{\delta(r-r_0)}{r^2} Y_1^{lm}(\theta_0, 0) F_r^{l2m} \\
 = \frac{2l+1}{4\pi \omega} \int \frac{\delta(r-r_0)}{r^2} d_1^{lm}(\theta_0) M^{r,-} \\
 = \left\{ \begin{array}{ll}
 \frac{2l+1}{4\pi \omega} \int \frac{\delta(r-r_0)}{r^2} M^{r,-} & \text{if } m = z_1 + z_2 \text{ and } l \geq |z_1 + z_2| \\
 0 & \text{otherwise}
\end{array} \right. \quad \text{[58]}
\]

To complete the evaluation of the coefficients \( f_0^{lm} \), corresponding to eqn [55], we need to find the spherical harmonic coefficients of the divergence of the field represented in eqn [58], for which we can employ eqn [51]; we find

\[
 f_0^{lm} = \left( \begin{array}{c}
 \left( F^r \right)_0^{lm} + \left( M^{r,-} \right)_0^{lm} - \left( M_{-0} \right)_0^{lm} + (M_{0-})_0^{lm} \\
 \left( F^r \right)_0^{lm} + \left( M^{r,-} \right)_0^{lm} - \left( M_{-0} \right)_0^{lm} + (M_{0-})_0^{lm} \\
 \end{array} \right) \quad \text{[59]}
\]

\[
 f_1^{lm} = \left( \begin{array}{c}
 \left( M^{r,-} \right)_0^{lm} - \left( M_{-0} \right)_0^{lm} + (M_{0-})_0^{lm} \\
 \left( M^{r,-} \right)_0^{lm} - \left( M_{-0} \right)_0^{lm} + (M_{0-})_0^{lm} \\
 \end{array} \right) \quad \text{[60]}
\]

\[
 \left( \begin{array}{c}
 \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 & 0 \\
 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 \\
 0 & 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) \\
 \end{array} \right) \quad \text{[61]}
\]

\[
 m = -2, l \geq 2
\]

\[
 \left( \begin{array}{c}
 \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 & 0 \\
 M_{00}^{r,-} \delta(r-r_0) & 0 & 0 \\
 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) \\
 \end{array} \right) \quad \text{[62]}
\]

\[
 m = -1, l \geq 1
\]

\[
 \left( \begin{array}{c}
 \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 & 0 \\
 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 \\
 M_{00}^{r,-} \delta(r-r_0) & 0 & 0 \\
 \end{array} \right) \quad \text{[63]}
\]

\[
 m = 0
\]

\[
 \left( \begin{array}{c}
 \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 & 0 \\
 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 \\
 0 & 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) \\
 \end{array} \right) \quad \text{[64]}
\]

\[
 m = 1, l \geq 1
\]

\[
 \left( \begin{array}{c}
 \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 & 0 \\
 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) & 0 \\
 0 & 0 & \frac{1}{\sqrt{2}} \delta(r-r_0) \\
 \end{array} \right) \quad \text{[65]}
\]

\[
 m = 2, l \geq 2
\]
These are the spherical harmonic coefficients of the force distribution [32] which will be needed in the following sections. The coefficients are zero for \(|m| > 2\).

### 1.03.4 The Green's Function for the Spherically Symmetric Earth

#### 1.03.4.1 Reduction of the General Equations to the Spherically Symmetric Case

We consider here the case in which the Earth model is spherically symmetric. In this case the equations of motion are separable in spherical coordinates, and thus can be solved by reduction to ordinary differential equations. Since deviations from spherical symmetry are relatively small in the Earth, they can subsequently be treated by perturbation theory. We assume that in the initial equilibrium configuration the stress is hydrostatic, that is

\[ t_{ij}^{0} = -\rho^{0} \delta_{ij} \]  \[ \text{[61]} \]

Spherical symmetry requires that \( r^{0} , \phi^{0} , \theta^{0} \) are functions only of \( r \). The gravitational acceleration is \( g_{0}^{0} = \delta_{ij} g^{0}(r) \delta_{ij} \), and the equilibrium equations [19], [20] and the boundary conditions in eqn [25] have the solutions:

\[ g^{0}(r) = \frac{4\pi G}{r^{2}} \int_{0}^{r} \rho^{0}(r') dr' \]

\[ \phi^{0}(r) = -\int_{r}^{\infty} g^{0}(r') dr' - \int_{0}^{r} g^{0}(r') dr' \]

\[ p^{0}(r) = \int_{r}^{\infty} \rho^{0}(r) g^{0}(r') dr' \]  \[ \text{[62]} \]

where \( a \) is the radius of the Earth and \( M \) is the Earth's total mass \( M = 4\pi \int_{0}^{a} r^{2} \rho^{0}(r) dr \). The equations of motion [26] with [17] and [20] can be put into the form

\[ \rho \left[ -\nabla^{2} u_{i} - u_{ik} g_{k} + \varphi_{,i} + (u_{ik} g_{k}) \right] = \left( C_{ijkl} u_{k} \right)_{,l} = f_{i} \]  \[ \text{[63]} \]

\[ \left( \varphi_{,k} + 4\pi G \rho u_{k} \right)_{,k} = 0 \]  \[ \text{[64]} \]

Where we have introduced the effective stiffness tensor

\[ C_{ijkl} = c_{ijkl} + \rho^{0} \left( \delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl} \right) \]  \[ \text{[65]} \]

which has the same symmetries (eqn [11]) as does \( c_{ijkl} \). In eqns [63], [64], and in subsequent equations, we drop the superscripts in \( \rho^{0} , g^{0} \), using simply \( \rho , g \) for these quantities. We also use the notation \( \varphi \) in place of \( \phi^{0} \). The applied force distribution \( f \), will be taken to be that given in eqns [32] and [55], having the spherical harmonic coefficients [60].

In a spherically symmetric model, the tensor field \( C_{ijkl} \) must be a spherically symmetric tensor field, and therefore its spherical harmonic expansion will have terms only of spherical harmonic degree \( l = 0 \). Its spherical contravariant components \( C^{a_{1} a_{2} a_{3} a_{4} }_{ijkl} = C^{a_{1} a_{2} a_{3} a_{4} }_{ijkl} \) must have indices that sum to zero, and must also satisfy the usual elastic tensor symmetries [14]. It is easily seen that there are only five independent components that satisfy these requirements: \( C^{0000} , C^{0001} , C^{0011} , C^{1111} , C^{001} + C^{111} \), which must be real (in the nonattenuating case) in order that the spherical components \( C_{ijkl} \) are real. Conventionally, these are designated (Love, 1927; Takeuchi and Saito, 1972)

\[
C^{0000} = C(r), \\
C^{* -00} = - F(r), \\
C^{+00} = - L(r), \\
C^{* -0} = 2 N(r), \\
C^{* -} = A(r) - N(r)
\]  \[ \text{[66]} \]

the independent spherical components being (using eqn [43])

\[
C_{ort} = C(r), \\
C_{ort\phi} = C_{ort\phi} = F(r), \\
C_{ort\theta} = C_{ort\theta} = L(r), \\
C_{ort\phi\phi} = N(r), \\
C_{ort\phi\theta} = A(r) - 2N(r), \\
C_{ort\phi\phi\phi} = A(r)
\]  \[ \text{[67]} \]

The mean bulk modulus \( \kappa \) and shear modulus \( \mu \) can be defined by

\[
\kappa = \frac{1}{9} C_{0000} = \frac{1}{9} \left( 4A + C - 4N + 4F \right), \\
\mu = \frac{1}{10} C_{0000} - \frac{1}{30} C_{0000} = \frac{1}{15} \left( A + C + 6L + 5N - 2F \right)
\]  \[ \text{[68]} \]

Other conventional notations are

\[
\lambda = \frac{2}{3} \mu, \\
u_{PV}^{2} = C / \rho, \\
u_{SH}^{2} = \lambda / \rho, \\
u_{SV}^{2} = L / \rho, \\
u_{SH}^{2} = N / \rho, \\
\eta = F / (A - 2L)
\]  \[ \text{[69]} \]

where \( \lambda \) (and \( \mu \)) are the Lamé parameters, \( v_{PV} \) and \( v_{SH} \) are the vertical and horizontal polarized compressional wave velocities, and \( v_{SV} \) and \( v_{SH} \) are the vertical and horizontal polarized shear wave velocities. Thus, radial anisotropy (also called transverse isotropy) is allowed in a spherically symmetric model. In fact, the PREM model, which is commonly used in normal mode calculations, has radial anisotropy in the top 220 km. This feature is essential to explain the frequencies of Love and Rayleigh waves simultaneously. In the case that the material is isotropic, \( A = C = \lambda + 2\mu = \kappa = \frac{4}{3} \mu, N = L = \mu, \) and \( \eta = 1 \).

#### 1.03.4.2 Derivation of the Radial Ordinary Differential Equations

Now we seek solutions \( u_{i} \varphi \) of eqns [63] and [64] in terms of a spherical harmonic expansions. It is convenient to write

\[
u_{im} = \nu_{i} \Omega_{l}(r) - iW_{im}(r)
\]  \[ \text{[70]} \]

\[
u_{0m} = \mu U_{lm}(r)
\]  \[ \text{[71]} \]

\[
u_{lm}^{*} = \nu_{i} \Omega_{l}(r) + iW_{im}(r)
\]  \[ \text{[72]} \]

where \( \nu_{i} = \sqrt{(2l + 1)} / 4\pi \), as the \( U, V, W \) notation is almost universally used in the literature on long-period seismology. \( U(r), V(r), \) and \( W(r) \) are the radial eigenfunctions which only depend on the radius. \( U(r) \) is used to describe motion in the radial direction, while \( V(r) \) and \( W(r) \) are the tangential scalars. The expansion corresponding to eqn [44] is then equivalent to the vector spherical harmonic representation (e.g., Morse and Feshbach, 1953):

---

where $Y_{lm}^m$ are the spherical harmonics $Y_{lm}$ as defined in eqn [41]. We shall abbreviate such vector spherical harmonic expansions using the shorthand

$$u_r = \sum_{lm} U_{lm}(r) v_1 Y_{lm}^0(\theta, \phi)$$

$$u_\theta = \sum_{lm} [V_{lm}(r) \partial_\theta + W_{lm}(r) \csc \theta \partial_\phi] v_1 Y_{lm}^0(\theta, \phi)$$

$$u_\phi = \sum_{lm} [V_{lm}(r) \csc \theta \partial_\phi - W_{lm}(r) \partial_\theta] v_1 Y_{lm}^0(\theta, \phi)$$

meaning that the vector field having Cartesian components $u_i$ is expressible in vector spherical harmonics as in eqn [73]. We shall also suppress the suffices $l, m$ and the explicit dependence upon $r$, writing simply $U, V, W$.

The spherical harmonic expansion of $f_i$ can also be converted into this vector spherical harmonic notation. Using eqn [60] with eqn [56] we obtain:

$$f_i \rightarrow \frac{v_i}{i \omega}$$

$$f_i := \sum_{lm} \frac{v_i}{i \omega} Y_{lm}^m(\theta, \phi)$$

Thus, ordinary differential equations in $r$ for $U, V, W, P, S, T$ are obtained by equating these to the forcing terms in eqn [75]. In addition, the expansion coefficients of the perturbation in gravitational potential, $\varphi = \varphi(r)$, are subject to equations derived from eqn [64]. The boundary conditions [25] require that $\varphi$ and $\partial_r \varphi + 4 \pi G \rho U$ are continuous throughout the model, and that $\varphi$ vanishes at infinity. For a given spherical harmonic degree $l$ the solutions of Laplace’s equation tending to zero at infinity are proportional to $r^{-l-1}$ and for this reason it is useful to define the new dependent variable $\psi = \partial_r \varphi + (l+1)$ continuous at interfaces between different regions of the model (from eqn [25]), it is usual to treat them as a new dependent variable, and to express the derivatives $\partial_r U, \partial_r V, \partial_r W$ in terms of them:

$$P = \frac{F r^{-1}}{}$$

$$S = \frac{L (\partial_r V + r^{-1} U)}{}$$

$$T = \frac{L (\partial_r W - r^{-1} V)}{}$$

where we have introduced traction scalars $P = P_{un}(r), S = S_{un}(r)$, and $T = T_{un}(r)$. Because the radial tractions are required to be

$$f_i := \left\{ \begin{array}{ll}
-\rho r^2 U + 2 r^{-2} (A - N)(2U - \zeta^2 V) + \rho (\partial_r g - 2 r^{-1} g) (U + \zeta^2 V) - \frac{1}{3} \rho g V - \partial_r \rho + 2 r^{-1} F \partial_r U - 2 r^{-1} P + \rho S + \rho \partial_r \varphi \\
-\rho r^2 V - r^{-2} A (2U - \zeta^2 V) + 2 r^{-2} N (U - V) + r^{-1} \rho (g V + \varphi) - \partial_r S - 3 r^{-1} S - r^{-1} \partial_r U \\
-\rho r^2 W - r^{-2} A (\zeta^2 V - 2) - \partial_r T - 3 r^{-1} T
\end{array} \right. $$

The vector spherical harmonic expansion of the left side of eqn [63] becomes:

$$f_i := \left\{ \begin{array}{ll}
-\rho r^2 U + 2 r^{-2} (A - N)(2U - \zeta^2 V) + \rho (\partial_r g - 2 r^{-1} g) (U + \zeta^2 V) - \frac{1}{3} \rho g V - \partial_r \rho + 2 r^{-1} F \partial_r U - 2 r^{-1} P + \rho S + \rho \partial_r \varphi \\
-\rho r^2 V - r^{-2} A (2U - \zeta^2 V) + 2 r^{-2} N (U - V) + r^{-1} \rho (g V + \varphi) - \partial_r S - 3 r^{-1} S - r^{-1} \partial_r U \\
-\rho r^2 W - r^{-2} A (\zeta^2 V - 2) - \partial_r T - 3 r^{-1} T
\end{array} \right. $$

Thus, the complete boundary value problem for $u(x, \omega)$ is to find, for each $l, m$, solutions $U_{lm}(r), V_{lm}(r), W_{lm}(r), \phi_{lm}(r), P_{lm}(r), S_{lm}(r), T_{lm}(r), \psi_{lm}(r)$, satisfying (i) equality of the
expressions in eqns [75] and [78], (ii) eqns [77] – in essence definitions of \( P, S, T \), and (iii) eqns [79] governing the self-gravitation.

The equations governing \( W \) and \( T \) are independent of the others, and so the problem naturally separates into the problems for the six functions \( U, V, \phi, P, S, \psi \), relating to spheroidal motion, and for the two functions \( W, T \), relating to toroidal motion. These are related to spheroidal and toroidal modes (see Figure 5). Spheroidal modes involve both radial and tangential motion, just like P-SV body waves. Toroidal modes only involve tangential motion. Examples of \( U, V, W \) for fundamental spheroidal and toroidal modes are given in Figure 6. In a fluid region, shear stresses are required to vanish, resulting in \( S = 0 \) and \( L = N = 0 \), \( A = C = P \); the second equation in [76] drops out and the equation arising from the second of eqn [78] can be solved for \( V, V = (\psi U - P + \varphi)/40^2 \), and thus \( S \) and \( V \) can be eliminated from the equations, resulting in equations for the four remaining variables \( U, \phi, P, \psi \). The case \( l = 0 \) leads to purely radial motion, \( V = 0 \) with \( \psi = \varphi/r \), \( d\varphi/dr = -4\pi G \psi U \), and the effective equations involve only \( U, \phi, P \). Examples of \( U \) for radial modes are given in Figure 7.

The equations are most conveniently stated as matrix differential equations, by rearranging them to give the radial derivatives of either the six (for spheroidal modes in a solid), four (for spheroidal modes in a fluid), or two functions (for radial and toroidal modes) to be determined in terms of the functions themselves. Here, we define stress-displacement vectors as follows:

\[
\text{Spheroidal solid: } \mathbf{y}^S = \begin{bmatrix} rU \\ rV' \\ r\phi \\ rP \\ rS \\ r\psi/4\pi G \end{bmatrix} \\
\text{Spheroidal fluid: } \mathbf{y}^f = \begin{bmatrix} rU \\ r\phi \\ rP \\ r\psi/4\pi G \end{bmatrix} \\
\text{Radial: } \mathbf{y}^R = \begin{bmatrix} rU \\ rP \end{bmatrix} \\
\text{Toroidal solid: } \mathbf{y}^T = \begin{bmatrix} rV' \\ rT \end{bmatrix}
\]

In each case, the resulting equations take the form:

\[
\frac{d\mathbf{y}}{dr} = A\mathbf{y} + a\delta(r-r_s) + b\delta'(r-r_l)
\]

where the vectors \( a = a(r, \omega, l, m) \), \( b = b(r, \omega, l, m) \) represent the forcing or earthquake source and can be readily derived from eqn [75]. The matrices \( A = A(r, \omega, l) \) can be written in terms of submatrices in the form

\[
A = \begin{pmatrix} T & K \\ S & -T \end{pmatrix}
\]

where \( K \) and \( S \) are symmetric and where \( T' \) is the transpose of \( T \). The fact that the equations have this special form stems from the fact that they arise from a variational principle, and in fact are a case of Hamilton’s canonical equations (Chapman and Woodhouse, 1981: Woodhouse, 1974). However, the usual variational derivations of the equations of motion neglect attenuation, and so it is interesting that this symmetry of the equations remains valid in the attenuating case. It plays an important part in methods of calculation normal modes (see the succeeding text), and also enables a complex version of the theory in attenuating media to be developed along the same lines. The specific forms of matrices \( T, K, S \) for spheroidal modes in solid regions are

\[
T^S = \begin{pmatrix} (1-2F/C)/r & \zeta F/C & 0 \\ -\zeta/r & 2/r & 0 \\ -4\pi G\rho & 0 & -l/r \end{pmatrix},
K^S = \begin{pmatrix} 1/C & 0 & 0 \\ 0 & 1/L & 0 \\ 0 & 0 & 4\pi G \end{pmatrix},
S^S = \begin{pmatrix} -\rho a_2 + 4(\gamma - \rho gr)/r^2 & 0 & 0 \\ \zeta (gr - 2\gamma)/r^2 - \rho (l + 1)/r & -\rho a_2 + [\zeta^2 (\gamma + N) - 2N]/r^2 & \rho c^2/r \end{pmatrix}
\]

In the case of radial modes, we also have \( \phi = 0 \) and \( \psi = 0 \), reducing the equations to

\[
T^R = \begin{pmatrix} (1-2F/C)/r & \zeta^2 /a_2 r^2 & 0 \\ -4\pi G\rho & -l/r & 0 \end{pmatrix},
K^R = \begin{pmatrix} 1/C & -\zeta^2 /a_2 r^2 & 0 \\ 0 & 1/L & 0 \end{pmatrix},
S^R = \begin{pmatrix} -\rho a_2 + 4(\gamma - \rho gr)/r^2 & 0 & 0 \\ \rho (a_2^2 /ro_2 - l - 1)/r & \rho c^2 /r^2 \end{pmatrix}
\]

In the case of toroidal modes, we are only solving for \( W \) and \( T \), leading to equations

\[
T^T = \begin{pmatrix} 2/r \\ 1/L \end{pmatrix},
S^T = \begin{pmatrix} -\rho a_2 + 4(\zeta^2 - 2N)/r^2 \end{pmatrix}
\]
boundary conditions at the center of the Earth and at the surface. This is an eigenvalue problem for \( \omega \) having solutions corresponding to the modes of free oscillation. The eigenvalues will be denoted by \( \omega_k \), where \( k \) is an index that incorporates the angular order \( l \), the overtone number \( n \) and the mode type: spheroidal or toroidal. The modal multiplets are conventionally given the names \( nS_l \) for spheroidals and \( nT_l \) for toroidals.

Overtone number is an index labeling the eigenfrequencies for a given \( l \) and for a given mode type, in increasing order. For radial modes, the overtone number equals the number of nodal lines along the radius (see Figure 7). Since the spherical harmonic order \( m \) does not enter into the equations, the modes are degenerate, meaning that there are \( 2l+1 \) different eigenfunctions, \( m = -l, -l+1, \ldots, l \) corresponding to the same given eigenvalue \( \omega_k \). The eigenfunctions will be denoted by \( s^{(km)}(x) \). These are the solutions \( u(x) \) given by eqn [73], for different values of \( m \), but with the same scalar eigenfunctions \( U(r), V(r), W(r) \). The set of \( 2l+1 \) eigenfunctions, \( s^{(km)}(x) \), for a given eigenfrequency \( \omega_k \) is said to constitute a multiplet. The eigenfunctions represent the spatial shape of a mode of free oscillation at frequency \( \omega_k \), because \( s^{(km)}(x)e^{i\omega_k t} \) is a solution of the complete dynamical equations in the absence of any forcing. Of course the eigenfunction is defined only up to an overall factor.

If the medium is attenuating, the eigenfrequencies will be complex, their (positive) imaginary parts determining the rate of decay of the mode with time. It is conventional to quantify this decay rate in terms of the ‘\( Q \)’ of the multiplet, \( Q_k \), which is defined in such a way that the mode decays in amplitude by a factor \( e^{-\pi/Q} \) per cycle. Therefore, \( Q_k = \text{Re} \omega_k/2\text{Im} \omega_k \), typically...
a large number, indicating that the modes decay by a relatively small fraction in each cycle. \( Q \) has also been called the quality factor.

In the case of forcing, eqn [81] leads to solutions \( y(l, m, r) \) which are discontinuous at the source radius \( r_s \). It can be shown that the discontinuity at \( r_s \) is given by (Hudson, 1969; Ward, 1980):

\[
[y]_{r=r_s} = s = a + Ab - \frac{db}{dr}
\]

and thus the boundary value problem for \( y \) requires the solution of the homogeneous equation \( dy/dr = Ay \) above and below the source, subject the conditions that the solution \( (i) \) is nonsingular at the center of the Earth, \( (ii) \) has vanishing traction components at the surface, and \( (iii) \) has the prescribed discontinuity \( s \) at the source radius \( r_s \). The specific forms for the discontinuity vector \( s \), using [75] and [87], are (As described earlier, we are here considering the source to be located at \( (\theta = \epsilon, \phi = 0) \), for some infinitesimal, positive \( \epsilon \); thus \( (F_{\theta}, F_{\phi}, F_r) \) coincide with \( (F_{\theta}, F_{\phi}, F_r) \) in the global Cartesian frame defined in eqn [37]. Similarly, \( (M_{\theta\theta}, M_{\phi\phi}, \text{etc.}) \) coincide with \( (M_{\theta\theta}, M_{\phi\phi}, \text{etc.}) \). If the source is located at a general colatitude \( \theta \) and longitude \( \phi_s \), the results can be applied in a rotated frame in which the \( (\theta, \phi, r) \) components map into \( \text{(South, East, Up)} \), coordinate \( \theta \) is epicentral distance and coordinate \( \phi \) is azimuth of the receiver at the source, measured anticlockwise from South.)

\[
\begin{align*}
\mathbf{s}^{r_s} &= \frac{v_s}{\omega r_s} \times \left[ \begin{array}{c}
\frac{r_s M_{rr}}{C} \\
0 \\
0 \\
-\frac{r_s F_r - M_{\theta\theta} - M_{\phi\phi} + 2M_{rr}F/C}{C} \\
\frac{\zeta (M_{\theta\theta} + M_{\phi\phi})/2 - \zeta M_{rr}F/C}{2} \\
0
\end{array} \right] \\
l \geq 1, m = 0
\end{align*}
\]

\[
\begin{align*}
\mathbf{s} &= \frac{v_s}{\omega r_s} \times \left[ \begin{array}{c}
\frac{r_s M_{rr}}{C} \\
0 \\
0 \\
0 \\
0 \\
\sqrt{\zeta^2 - 2(-M_{\theta\theta} + M_{\phi\phi})/4}
\end{array} \right] \\
l = m = 0
\end{align*}
\]

\[
\begin{align*}
\mathbf{s}^{r_s} &= \frac{v_s}{\omega r_s} \times \left[ \begin{array}{c}
\frac{r_s M_{rr}}{C} \\
\frac{\zeta (M_{\theta\theta} - M_{\phi\phi})/2}{2} \\
\frac{r_s (\mp M_{\theta\theta} + iM_{\phi\phi})/2L}{2} \\
\frac{r_s (\mp F_{\theta} + iF_{\phi})/2}{2} \mp \frac{M_{\theta\theta} - M_{\phi\phi} - iM_{rr}}{4}
\end{array} \right] \\
l \geq 1, m = 0
\end{align*}
\]

\[
\begin{align*}
\mathbf{s} &= \frac{v_s}{\omega r_s} \times \left[ \begin{array}{c}
\frac{r_s M_{rr}}{C} \\
0 \\
\frac{\zeta (M_{\theta\theta} - M_{\phi\phi})/2}{2} \\
\frac{r_s (\mp M_{\theta\theta} + iM_{\phi\phi})/2L}{2} \\
\sqrt{\zeta^2 - 2(-iM_{\theta\theta} + M_{\phi\phi})/4}
\end{array} \right] \\
l \geq 1, m = \pm 1
\end{align*}
\]

\[
\begin{align*}
\mathbf{s}^{r_s} &= \frac{v_s}{\omega r_s} \times \left[ \begin{array}{c}
\frac{r_s M_{rr}}{C} \\
0 \\
\frac{\zeta (M_{\theta\theta} - M_{\phi\phi})/2}{2} \\
\frac{r_s (\mp M_{\theta\theta} + iM_{\phi\phi})/2L}{2} \\
\sqrt{\zeta^2 - 2(-iM_{\theta\theta} + M_{\phi\phi})/4}
\end{array} \right] \\
l \geq 2, m = \pm 2
\end{align*}
\]

where the elastic parameters are those evaluated at the source radius \( r_s \). We consider here only the case that the source is located in a continuous, solid region of the model.
1.03.5 Numerical Solution of the Radial Equations*

1.03.5.1 Methods of Solution*

The inhomogeneous (i.e., with forcing term $f_j$) boundary value problem as formulated above gives a unique solution for each value of $\omega$. The solution in the time domain can then be obtained in the form of an integral in $\omega$, using the inverse Fourier transform [28]. This, in essence, is the basis of several practical methods for calculating theoretical seismograms, for example, the reflectivity method (Fuchs and Muller, 1971) and the direct solution method of Friederich and Dalkolmo (1995). Alternatively, the inverse transform can be evaluated by completing the integration contour in the upper half of the complex $\omega$ plane. Then it is found that the solution can be reduced to a sum over residues, each pole of the integrand corresponding to a particular mode of free oscillation of the model. A more usual approach to the normal mode problem is to consider first the free (i.e., unforced) modal solutions of the equations, and then to make use of the orthogonality and completeness properties of the eigenfunctions to obtain solutions of the inhomogeneous (i.e., forced) problem. Here, we have examined first the inhomogenous problem because the demonstration of orthogonality and completeness in the general (attenuating) case is a nontrivial issue, and it is only by virtue of the analysis of the inhomogeneous problem, and the resulting analytic structure of the integrand in the complex $\omega$ plane, that orthogonality and completeness can be demonstrated. It is necessary to show (in the nonattenuating case) that the only singularities of the integrand are simple poles on the real $\omega$-axis. Then the modal sum emerges and completeness is demonstrated by the solution itself. In the attenuating case the situation is more complex, and there are other singularities located on the positive imaginary $\omega$-axis – let us call them the relaxation singularities, as they are associated with decaying exponential functions in the time domain. Thus, while the solution developed here for the inhomogeneous problem remains valid in this case, arguments based on orthogonality and completeness cannot be made. The solution can nevertheless be derived in the form of a sum over residues, and other singularities. While the contribution from relaxation singularities is the main focus of attention in the analysis of post-seismic excitations, they are expected to make negligible contributions for the typical seismic application. However, even in the seismic domain it is necessary to know modal excitations in the attenuating case, and these are difficult to determine, other than by a rather complex application of mode-coupling theory (Lognonne, 1991; Tromp and Dahlen, 1990), which will be difficult to carry out to high frequencies. Using the inhomogeneous solution, on the other hand, the ‘seismic’ modes and their excitations emerge naturally as the contributions from the residues of poles near the real axis, and can be calculated exactly and economically.

In both the attenuating and nonattenuating cases and for both the homogeneous and inhomogeneous problems, the integration of the ordinary differential equation presents severe numerical difficulties. One problem is that the equations are such that evanescent – exponentially increasing and decreasing – solutions exist on more than one spatial scale. At moderately high frequencies, when the equations are integrated numerically, the solutions are effectively projected onto the solution having the most rapid exponential increase, and thus even though a linearly independent set of solutions is guaranteed, theoretically, to remain a linearly independent set, it becomes, numerically, a one-dimensional projection. The general solution of this difficulty is to reformulate the equations in terms of minors (i.e., subdeterminants) of sets of solutions (Gilbert and Backus, 1966). The standard method for normal mode calculations (Woodhouse, 1988) is based on the minor formulation of the equations, and uses a novel generalization of Sturm–Liouville theory to bracket modal frequencies, itself a nontrivial issue for the spheroidal modes, as the modes are irregularly and sometimes very closely spaced in frequency, making an exhaustive search difficult and computationally expensive. The program MINOS of Guy Masters, developed from earlier programs of Gilbert, and of Woodhouse (also a development of Gilbert’s earlier programs), implements this method, and has been generously made available to the community. The direct solution method for the inhomogeneous problem of Friederich and Dalkolmo (1995) is based on the minor formulation in the non-self-gravitating case, developed for the flat-earth problem by Woodhouse (1980b). Here, we outline some of the key features of the minor approach.

1.03.5.2 Numerical Integration*

Let us be specific by assuming that the model has a solid inner core, a fluid outer core, and a solid mantle. It may or may not have an ocean. Let us also consider the case of spheroidal oscillations, for which the solution vector is six dimensional in solid regions ($U^S$) and four dimensional in fluid regions ($V^F$). Toroidal and radial modes are much simpler. The basic method of solution is to start at the center of the Earth, and to specify that the solution should be nonsingular there. By assuming that the medium is homogeneous and isotropic within a small sphere at the center, it is possible to make use of known analytical solutions in terms of the spher-4ical Bessel functions (Love, 1911; Pekeris and Jarosch, 1958; Takeuchi and Saito, 1972). Thus (in the spheroidal case that we are considering), there is a three-dimensional set of solutions to be regarded as candidates for components of the solution at the center. Using these three solutions as starting solutions, the equations can be integrated toward the surface, for example, using Runge–Kutta techniques. We introduce a $6 \times 3$ matrix $Y = Y^{FS}(r)$ having columns equal to these three solutions, and which has $3 \times 3$ subpartitions $Q = Q^{FS}(r)$ and $P = P^{FS}(r)$, that is, $Y = \begin{bmatrix} Q & \end{bmatrix}$. What is important about $Y$ is the subspace of six-dimensional space that is spanned by its three columns, a property that is left unchanged if $Y$ is post-multiplied by any nonsingular $3 \times 3$ matrix. Assuming, for the moment, that $Q$ and $P$ are nonsingular, and post-multiplying by $Q^{-1}$ and by $P^{-1}$ we conclude that both $\begin{bmatrix} Q & U \end{bmatrix}$ and $\begin{bmatrix} V & 1 \end{bmatrix}$, where $V$ and $U$ are the mutually inverse matrices $V = QP^{-1}$, $U = PQ^{-1}$ and where $1$ is the unit matrix, have columns spanning the same three-dimensional space as is spanned by the columns of $Y$. An unexpected property of $U$, $V$, stemming from the self-adjointness property of the equations and boundary conditions, is that

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*Starred sections are intended for the more specialist reader, and the chapter can be understood without reading these extra sections.
they are symmetric. It is not difficult to show that by virtue of the particular structure of the differential equations noted in eqn \[82\] that if \(\mathbf{U}\), and (therefore) \(\mathbf{V}\) are symmetric at a given radius, then they remain symmetric as the equations are integrated to other radii. To demonstrate this, consider:

\[
\frac{d}{dr}(\mathbf{Q}^{P}P^{Q}) = \frac{d}{dr}(\mathbf{Y}^{\Sigma}Y) = \mathbf{Y}^{A}Y^{\Sigma} + Y^{\Sigma}Y^{\Lambda} = 0
\]

where we have introduced the partitioned matrix

\[
\mathbf{\Sigma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

where 1 is the unit matrix of appropriate dimension, so that the matrix

\[
\mathbf{\Sigma} \mathbf{A} = \begin{pmatrix} \mathbf{S} & -\mathbf{T} \\ \mathbf{T} & \mathbf{K} \end{pmatrix}
\]

is symmetric. We shall also make use of the fact (Woodhouse, 1988) that in the nonattenuating case, \(-\mathbf{\Sigma}\mathbf{A}_{l}\), by which we shall mean the partial derivative of \(\mathbf{A}\) with respect to \(x = \varepsilon \gamma^{2}\), is positive semidefinite, that is, that \(y^{\prime}(-\mathbf{\Sigma}\mathbf{A}_{l})y \geq 0\) for any real column \(y\), as can be verified using the forms for the submatrices of \(\mathbf{A}\) \[83\]–\[86\]. Thus, if \(\mathbf{Q}^{P}P^{Q}\) vanishes at a given radius, it vanishes everywhere in the interval over which the equations are being integrated. But we can write \(\mathbf{Q}^{P} - \mathbf{P}^{Q} = \mathbf{Q}^{P} (\mathbf{P}^{-1} - \mathbf{Q}^{-1}) \mathbf{P}^{Q}\), which shows that \(\mathbf{P}^{-1}\) is symmetric if \(\mathbf{Q}^{P} - \mathbf{P}^{Q}\) vanishes. It is interesting to note that this argument does not rely on \(\mathbf{Q}\) and \(\mathbf{P}\) being nonsingular throughout the interval of integration, since \(\mathbf{Q}^{P} - \mathbf{P}^{Q}\) remains finite and continuous. It is a lengthy algebraic exercise to show that \(\mathbf{U}\) and \(\mathbf{V}\) are symmetric at the center of the Earth (i.e., when the analytic solutions are used), but nevertheless this can be verified (it can be easily checked numerically).

To continue the narrative, the equations for \(Y^{C}\) are being integrated in the inner core, and we arrive at the inner core boundary. Here, the component of the solution corresponding to the shear traction on the boundary (the fifth element in our notation) is required to vanish. Thus, at the boundary we need to select from the three-dimensional space spanned by the columns of \(Y\) the (in general) two-dimensional subspace stress-displacement vectors having vanishing fifth elements. This subspace is most easily identified by considering the basis constituted by \(\begin{bmatrix} \mathbf{V} \\ 1 \end{bmatrix}\), since its first and third columns have vanishing fifth elements, and thus by deleting the middle column, together with the second and fifth rows, as they correspond to variables not needed on the fluid side, we obtain the following rule for transmitting the basis of allowable solutions from the solid to the fluid side of the boundary:

\[
\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} u_{11} & u_{13} \\ u_{13} & u_{33} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

We can now continue the integration, using the \(4 \times 2\) matrix \(Y^{OC}\) in the fluid outer core. We can define \(\mathbf{Q}, \mathbf{P}, \mathbf{V}, \mathbf{U}\) similarly, now \(2 \times 2\), rather than \(3 \times 3\) matrices, and again \(\mathbf{V}, \mathbf{U}\) are symmetric. Continuing the integration, we arrive at the outer core boundary, and again need to consider how to transmit the solution space across the boundary. Elements in rows 1, 2, 3, 4 on the fluid side need to be continuous with elements in rows 1, 3, 4, 6 on the solid side. The fifth element on the solid side, the shear traction component \(r_{5}\), has to vanish. Since the horizontal displacement can be anything on the solid side we have to add to the basis to represent solutions having nonvanishing horizontal displacements on the solid side. The easiest way to satisfy these requirements is to consider the basis represented by \(\begin{bmatrix} 1 \\ \mathbf{U} \end{bmatrix}\). It can be easily verified that the following rule satisfies the requirements:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_{11} & 0 & u_{12} \\ 0 & 0 & u_{22} \end{pmatrix}
\]

The middle column has been inserted to represent the fact that the solution space has to contain vectors with nonvanishing horizontal displacement scalar \((r_{5}V)\) at the base of the mantle.

We can now continue the integration through the mantle. In the inhomogenous problem, the next interesting event is when we arrive at the source radius \(r_{s}\). In the homogeneous problem, on the other hand, we can continue the integrations to the ocean floor, applying the same rule as at the inner core boundary to transmit the solution space into the ocean, and then continue to the surface, where the free surface condition needs to be satisfied. The requirement, of course, is that there be a linear combination of the columns of the \(4 \times 2\) matrix \(Y^{O} = \begin{bmatrix} Q^{O} \\ P^{O} \end{bmatrix}\) (where superscript ‘\(O\)’ is for ocean) that have vanishing surface traction scalar \(r_{1}\) and vanishing gravity scalar \(r_{1}/4\pi G\), that is, vanishing elements 3 and 4. This requires \(\det(P^{O}) = 0\) at the free surface. In the absence of an ocean, we similarly need \(\det(P^{O}) = 0\) where \(P^{C}\) is the \(3 \times 3\) matrix in the crust.

In the inhomogeneous case, we need to arrange for there to be a prescribed discontinuity \(s^{h}\) at the source depth. Thus, we need to characterize the solution space above, as well as below the source. At the surface of the ocean the solution must have vanishing elements 3 and 4, and thus we choose \(Y^{O}\) at the free surface to be, simply \(Y^{O} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\). Now we can integrate the solution downwards, using the same fluid–solid rule at the ocean floor as was employed at the core–mantle boundary, until we reach the source radius \(r_{s}\) from above. If \(\begin{bmatrix} 1 \\ \mathbf{U}_{1} \end{bmatrix}\) spans the solution space below the source and \(\begin{bmatrix} 1 \\ \mathbf{U}_{2} \end{bmatrix}\) spans the solution space above the source, we need to solve

\[
\begin{bmatrix} 1 \\ \mathbf{U}_{2} \end{bmatrix} x_{2} - \begin{bmatrix} 1 \\ \mathbf{U}_{1} \end{bmatrix} x_{1} = s^{h}
\]

for the two 3-vectors \(x_{1}, x_{2}\) which represent the multipliers for the columns of each matrix that are needed to satisfy the
condition that the solution have the prescribed discontinuity. The solution is easily found to be

\[ x_1 = (U_2 - U_1)^{-1} \left( s_{Ss}^u - U_2 s_{Ss}^t \right), \]

\[ x_2 = (U_2 - U_1)^{-1} \left( s_{Sp}^u - U_1 s_{Sp}^t \right) \]

where \( s_{Ss}^u, s_{Sp}^u \) are the upper and lower halves of \( s_{Ss}^s \), and the solution vectors below and above the source are given by

\[ y_1^S = \begin{bmatrix} x_1 \\ U_1 x_1 \end{bmatrix}, \quad y_2^S = \begin{bmatrix} x_2 \\ U_1 x_2 \end{bmatrix} \]

This determines the linear combination of the basis vectors that are needed to satisfy the source discontinuity condition, and hence to determine the solution at any point of the medium and, in particular, at the surface, where it may be required to calculate some seismograms.

### 1.03.5.3 Eigenfunctions and Eigenfrequencies

There will be singularities in the integrand of the inverse Fourier transform when \( U_2 - U_1 \) is singular. This will occur for frequencies \( \omega \) for which a solution exists to the homogenous problem, that is, at the frequencies of free oscillation. If \( U_2 - U_1 \) is singular at a particular source radius it is, therefore, necessarily singular at all radii. To evaluate the inverse transform as a sum over residues, we can write,

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)e^{i\omega t} d\omega = \sum_{k} \lim_{\omega \to \omega_k} \frac{\Delta(\omega)I(\omega)}{\Delta'(\omega_k)} \]

where \( \omega_k \) is a mode frequency and where \( \Delta(\omega) = \det I(\omega) \) is that factor in the denominator of the integrand \( I(\omega) \) that vanishes at \( \omega_k \), assuming that it has a simple zero at \( \omega_k \). Thus, we can replace the inverse transform by a sum over residues provided that the singular part of the integrand is replaced by the expression corresponding to it on the right side of eqn [97]. From eqn [95], we find that the necessary replacement is

\[ (U_2 - U_1)^{-1} \left[ \frac{\partial \text{adj}(U_2 - U_1)}{\partial x} \right]_{x = -\omega_k} = \frac{1}{2\omega_k} z_k z_k' \]

where \( \text{adj} \) represents matrix adjoint – the matrix of cofactors. The second equality defines the column \( z_k \) and its transpose \( z_k' \), and arises from the fact that the adjoint of a singular matrix, assuming that the rank defect is 1, is expressible as a dyad; the factor \(-1/2\omega_k\) is included in the definition for convenience, as with this definition of \( z_k \), it can be shown that the column

\[ y_k = \begin{bmatrix} z_k \\ U_1 z_k \end{bmatrix} = \begin{bmatrix} z_k \\ U_1 z_k \end{bmatrix} \]

is an eigenfunction (i.e., a solution of \( \text{dy/dr} = Ay \)) and has normalization

\[ \int_0^1 y_k^* (-\Sigma A) y_k dr = 1 \]

where the notation \( -\Sigma A \) is that introduced in the discussion following eqn [82]. The eigenfunctions can be found without needing to calculate the derivatives of solutions with respect to \( \omega_k \) as it can be shown that \( \det Q_i \det Q_j \det \det(U_2 - U_1) \) is independent of \( r \). This is the basis for the construction of the eigenfunctions from solutions of the minor equations, although it was arrived at differently in Woodhouse [1988]. We see that it is necessary to integrate both upwards and downwards in order to obtain \( U_1 \) and \( U_2 \) (or, rather, the minors from which they can be constructed, see below) at all radii \( r \).

In the nonattenuating case, the eigenfunctions \( y_k \) are real, and eqn [99] reduces to the standard normalization conditions for the scalar eigenfunctions \( U_1, V_1, W_1 \) (defined in terms of \( y_k \) as in eqn [80]).

\[ \text{spheroidal: } \int_0^1 \rho (U_1^2 + \xi^2 V_1^2) r^2 dr = 1, \]

\[ \text{toroidal: } \int_0^1 \rho^2 r^4 W_1^2 dr = 1 \]

In the attenuating case, on the other hand, the eigenfunctions are complex and the normalization condition includes terms arising from the derivatives of the elastic parameters with respect to \( \omega \). In this case, eqn [99] determines both the phase and the amplitude of the eigenfunction. Using the replacement [98] in [95], and making use of the definitions of \( s_{Ss}^s \), and similarly \( s_{Sp}^s \), from eqns [88] to [90] it can be shown that the inhomogeneous solution, now in the time domain, can be written as a sum over residues:

\[ y_{in}(r, t) = \sum_k \frac{1}{2\omega_k} E_{\text{am}} y_k(r) e^{i\omega t} \]

where modal excitations \( E_{\text{am}} \) are given by

\[ E_{\text{am}} = \begin{cases} y_k^S s_{Sp}^S \text{ spheroidal} \\ y_k^T s_{Sp}^T \text{ toroidal} \end{cases} \]

\[ \begin{aligned} &\frac{1}{\tau_1} \left[ -r_1 U_1 F_r + \tau_1 \partial_r U_1 M_{\tau r} + U_1 (M_{\theta\theta} + M_{\phi\phi}) + \frac{1}{2} \tau_1^2 V_1 (M_{\theta\theta} + M_{\phi\phi}) + \frac{1}{2} \tau_1^2 W_1 (M_{\theta\theta} - M_{\phi\phi}) \right] m = 0 \\ &\frac{1}{\tau_1} \left[ -r_1^2 (V_1 \mp i W_1) (\pm F_1 - i F_{\phi}) + \frac{1}{2} \tau_1 (\partial_r V_1 + i \partial_r W_1) (\mp M_{\theta\theta} - i M_{\theta\phi}) + \frac{1}{2} \tau_1 (U_1 - V_1 \pm i W_1) (\pm M_{\phi \theta} - i M_{\theta \phi}) \right] m \geq 1, m = \pm 1 \\ &\frac{1}{4} \tau_1 \left[ -r_1^2 (V_1 \mp i W_1) (M_{\theta\theta} - M_{\phi\phi} \pm i M_{\theta\phi} \pm i M_{\theta \phi}) \right] m \geq 2, m = \pm 2 \end{aligned} \]

where eqn [76] has been used to express the radial traction components \( P_0, S_0, T_0 \) in \( y_k \) in terms of \( U_1, V_1, W_1 \) and their derivatives. The eigenfunctions are those evaluated at the source radius \( r_c \). In eqn [102], we have combined the results for toroidal and spheroidal modes; of course, for spheroidal modes \( W_1 = 0 \) and for toroidal modes \( U_1 = V_1 = 0 \). This result for the modal excitations is equivalent, in the case of a symmetric moment tensor to the forms given in Table 1 of Woodhouse and Gimius (1982).

The sum over residues [101] needs to be carried out over all simple poles in the upper half of the complex \( \omega \)-plane. It will include the oscillatory ‘seismic modes’ having \( \text{Re} \omega_k \neq 0 \), and

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The seismic modes will occur in pairs, with a partner, obtained by reflection in the imaginary axis, at \(-\omega^*\). It is not necessary to consider the modes for which \(\text{Re} \omega < 0\) explicitly. As a result, the final result for the displacement in the time domain is calculated, it is possible to include them automatically by adding the complex conjugate, in order that the final result should be real. In the attenuating case, the solution is still not necessarily complete, as the constitutive law may introduce a branch cut along the positive imaginary axis, corresponding to a continuous distribution of relaxation mechanisms. In this case the sum over residues needs to be augmented by an integral around any branch-cut singularities on the positive imaginary axis. In order to include the ‘static’ response of the medium it is necessary to analyze the behavior of the integrand at \(\omega = 0\). We do not pursue this in detail here but make some general observations. Because the source that is being considered has, in general, nonvanishing net force and moment, we would obtain secular terms corresponding to translational and rotational rigid motions (degree \(l = 1\) spheroidal and toroidal modes having zero frequency). If the force is set to zero and the moment tensor is taken to be symmetric, these modes would not be excited. In this case, the final displacement field, after all seismic modes and relaxation modes have died away can be found from the residue at zero frequency, and will correspond to the static \((\omega = 0)\) solution of the equations in which the elastic parameters are replaced by their values at zero frequency – their so-called relaxed values. Alternatively, the final displacement can be found by considering the fact that if the static terms are omitted the final displacement is zero, because all modes attenuate with time, whereas, in fact, it is the initial displacement that should be zero. Thus, the static terms must be such as to cancel the dynamic terms at zero time. It is not obvious that these two different ways of evaluating the static response will agree (i.e., using the static solution for relaxed values of the moduli, or using the fact that the initial displacement must be zero). We conjecture, but do not claim to prove, that these two methods will give the same result. If this is so, it means that, provided that all modes are included in the sum, we can include the static response by substituting \(e^{\text{inh}t} - 1\) for \(e^{\text{inh}t}\) in eqn [101] – that is, by subtracting the value at zero time.

### 1.03.5.4 Minors*

A central role is played in the foregoing theory by the symmetric matrices \(U\) and \(V\), notwithstanding that they possess singularities within the domain of integration. From their definitions \(V = Q^{-1}P^{-1}\), \(U = P^{-1}Q^{-1}\), using the following standard formula for the inverse of a matrix in terms of its cofactors and its determinant, they can be expressed in terms of the various \(3 \times 3\) or \(2 \times 2\) subdeterminants of \(Y\). Explicitly we have:

\[
U = \frac{m_2}{m_1} \quad V = \frac{m_1}{m_2} \quad (n = 1) \quad [103]
\]

where \(m_n\) are the minors of the relevant \(2 \times 1, 4 \times 2, 3 \times 6\) \((n = 1, 2, or 3)\) solution matrices \(Y\). We are using a standard way of enumerating these (see Gilbert and Backus, 1966; Woodhouse, 1988). We include the \(n = 1\) case here, which is relevant to the case of toroidal and radial oscillations, for the sake of completeness. The results still hold in this case, although the matrices \(Q, P, V, U\) reduce in this case to simple numbers, and the minors reduce to the elements of the solution vector itself (the \(1 \times 1\) subdeterminants of a \(2 \times 1\) matrix \(Y\)). It is well known (Gilbert and Backus, 1966) that differential equations can be derived that are satisfied by the minors, and thus that they can be calculated directly, without the need to integrate the equations for particular solution sets \(Y\). Thus, formulae and results involving \(V, U\) can be readily transcribed into formulæ involving the minors. Essentially, any one of \(U, V, m\) provides a way of characterizing a subspace of the \(2n\) dimensional space of interest \((n = 1, 2, or 3)\), in a way that is independent of any specific basis. However, the minors have the practical advantage that they do not become infinite in the domain of integration.

The matrices \(U\) and \(V\) possess another remarkable property which results from the positive semi-definiteness of \(-\Sigma A\) in the nonattenuating case. Using this property, it is possible to show that the derivatives \(U_i\) and \(V_i\) also have definiteness properties. For upward integration \(U_i \leq 0\) and \(V_i \geq 0\) where \(\geq 0\) and \(\leq 0\) is used as a shorthand for the relevant semi-definiteness property. To prove this, consider the matrix \(P^{-1}Q - Q^{-1}P = -U\Sigma U_i\). The radial derivative of this matrix is given by

\[
\frac{\text{d}}{\text{d}r}(P^{-1}Q_i - Q^{-1}P_i) = -\frac{\text{d}}{\text{d}r}(\Sigma Y_i)
\]

\[
= -(Y\Sigma^\prime Y_i) - \Sigma (A_iY + AY_i)
\]

\[
= -\Sigma A_iY \geq 0 \quad [106]
\]

where we have used \(\text{d}Y/\text{d}r = AY\), together with its transpose and its derivative with respect to \(\lambda\). The cancelation of the terms not involving \(A_i\) is due to the symmetry of \(\Sigma A\) and the antisymmetry of \(\Sigma\). We also have

\[
V_i = (QP^{-1})_i = Q_iP^{-1} - Q^{-1}P_iP^{-1} = Q_iP^{-1} - P^{-1}Q_iP^{-1}P^{-1}
\]

Thus, eqn [106] shows that if \(P^{-1}Q_i - Q^{-1}P_i \geq 0\) at some initial point, then it remains positive semi-definite during upward integration. Then, from [107], \(V_i \geq 0\), as we wished to show. Using the analytic solutions at the center of the Earth, it can be shown that \(V_i\) does have the required properties at the starting point of integration, being independent of frequency at the
center of the Earth. Also, its semidefiniteness property is preserved on passing from solid to fluid and vice versa. Similarly, it can be shown that for upward integration $U_L \leq 0$.

The diagonal elements of $V_L$ and $U_L$, which necessarily share the semi-definiteness properties of the matrices themselves, require, for upward integration, that the diagonal elements of $V$ and $U$ are nondecreasing and nonincreasing functions of frequency, respectively. As a function of frequency these diagonal elements behave like the familiar tangent and cotangent function, having monotonic increase or decrease between their singularities. The singularities in $V$ at the surface are of particular interest, since the frequencies for which $V$ is singular (i.e., infinite) at the surface are precisely the frequencies of the normal modes. One particular diagonal element, namely $v_{11}$ in the notation used here, has the additional property, using [92], [93], that it is continuous at solid–fluid and fluid–solid boundaries. In the case of fluid–solid boundaries this is not so obvious, as both $m_{110}$ and $m_{210}$ vanish on the solid side ($v_{11} = m_{110}/m_{210}$ eqn [105]), but it can be shown that the limit as the boundary is approached from the solid side is, in fact, equal to $v_{11}$ on the fluid side.

1.03.5.5 Mode Bracketing and Counting

The function $\theta_R(r, \lambda) = -\pi \cot^{-1}(v_{11})$, which can be made continuous (as a function of $r$ and as a function of $\lambda$) through singularities of $v_{11}$, has the properties that (i) it is independent of frequency at the center of the Earth, (ii) it is nondecreasing as a function of $\lambda$, and (iii) it takes on integer values at the surface at the frequencies $\omega^2 = \lambda$ corresponding to the normal modes. This makes it an ideal mode counter, since two integrations of the equations, at frequencies $\omega_1$, $\omega_2$, say, can determine the values $\theta_R(a, \omega_1^2)$, $\theta_R(a, \omega_2^2)$ at the Earth’s surface and it is necessary only to find how many integers lie between these values to determine how many modal frequencies lie between $\omega_1$ and $\omega_2$. There is a complication associated with fluid–solid boundaries. As discussed above, both $m_{110}$ and $m_{210}$ vanish on the solid side, even though $v_{11}$ remains continuous. This circumstance leads to singular behavior of $\theta_R$ as a function of $\lambda$, and we find that it is necessary to increment $\theta_R$ by 1 at a fluid–solid boundary when the $(2,2)$ element of $S^{**}$ (eqn [83]) is negative on the solid side, that is, for $\omega^2 > [\cos^2(\lambda - \lambda') - 2]/\rho r^2$, in which the elastic constants are those evaluated at the boundary, on the solid side and $r$ is the radius of the boundary. For upward integration (the usual case), this occurs at the core–mantle boundary.

Figure 8 shows an example of the behavior of $\theta_R(x, \lambda)$ for spherical oscillations of degree $l = 10$. The $\theta_R$ mode counter can be used to bracket the modal frequencies by a bisection method that seeks values of frequency such that $\theta_R(a, \omega_s^2)$ takes on values lying between any pair of successive integers in an interval $[\theta_R(a, \omega_{s_{\min}}), \theta_R(a, \omega_{s_{\max}})]$.

Having bracketed the modal frequencies (for a given $l$) it is necessary to converge on the zeros of $\det[P]_{\nu\nu}$. This can be done in a variety of standard ways, bisection being the ultimately safe method if all else fails. Figure 9 shows the resulting dispersion diagram for spheroidal modes up to 30 mHz. The crowding and irregularity of the distribution in the left side of the diagram demonstrate the need for the mode-counting scheme. For the toroidal modes, the dispersion diagram is much simpler (Figure 10), and so the mode-counting scheme is less critical.

1.03.6 Elastic Displacement as a Sum Over Modes

We shall here assume that a catalog of normal mode eigenfrequencies and scalar eigenfunctions has been calculated, including spheroidal modes (Figure 9) and toroidal modes.

Figure 8 An example of the behavior of $\theta_R(x, \lambda)$ for the spherical oscillations of degree $l = 10$. It has been calculated for the true modal frequencies up to $20S_{10}$, and thus the surface value of $\theta_R(x, \lambda)$, at normalized radius $x = 1$, takes on successive integer values equal to the overtone number. $\theta_R$ can be seen to be independent of frequency at the center of the Earth ($x = 0$) where $\theta_R(0, \lambda) = 1/2$. The nondecreasing property of $\theta_R$ as a function of $\lambda = \omega^2$ means that successive curves never cross. Notice that $\theta_R$ is not a monotonic function of $r$. The discontinuity in $\theta_R$ at the core–mantle boundary, discussed in the text, affects the mode count for modes higher than $20S_{10}$. In a sense, the values of $\theta_R(x, \lambda) = -(1/\pi)\cot^{-1}v_{11}$ are not of physical significance, since $v_{11}$ is a dimensional quantity, and thus the value of $\theta_R(x, \lambda)$ depends on the units employed. In any system of units, however, $\theta_R$ takes on integer values at the same values of its arguments. In other words, it is the zero crossings of $v_{11}$, for which $\theta_R$ acts as a counter, that are of primary significance. The results shown here are for $\theta_R(x, \lambda) = -(1/\pi)\cot^{-1}20v_{11}$, with $x = 4 \times 10^7$ N m$^{-2}$. 

Rather than pursuing the inhomogenous problem outlined in earlier sections, we shall now adopt the more classical approach, making use of the orthogonality properties of the eigenfunctions to expand the applied force distribution and to find the modal excitations. The argument is strictly valid only in the nonattenuating case. Attenuation is subsequently included by introducing decay constants associated with each mode. Recall that the eigenfrequencies $\omega_k$ and the eigenfunctions $s^{(km)}$ are solutions of the eigenvalue problem

$$\mathcal{H} s^{(km)} = \rho \omega_k^2 s^{(km)} \quad [108]$$

where each multiplet is labeled by $k = n, l, q$ with $q$ denoting the modal type (either spheroidal or toroidal), $l$ the angular order and $n$ the overtone number. $m$ is the azimuthal order. The problem is degenerate in the spherical nonrotating case, so for each normal mode multiplet there are $2l+1$ eigenfunctions $s^{(km)}$ (labeled by $m$), all with the same eigenfrequency $\omega_k$. It can be shown that the operator $\mathcal{H}$ is self adjoint in the sense

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**Figure 9** The dispersion diagram for spheroidal modes to 30 mHz. Modes with the same overtone number $n$ are connected by lines, forming branch. The fundamental branch ($n = 0$) and the first overtone branch ($n = 1$) are indicated. Also shown are the Stoneley branches, which are modes that only exist at solid–fluid interfaces.

**Figure 10** The dispersion diagram for toroidal modes to 30 mHz. Modes with the same overtone number $n$ are connected by lines, forming branch. The fundamental branch ($n = 0$) and the first overtone branch ($n = 1$) are indicated.
harmonics $Y_l^m(\theta, \phi)$. Thus, it is a further exercise in spherical harmonic analysis to reduce the excitation coefficients, $E_{l m} = (F_{l m} (x_i))^* + M_{l m} (x_i))^*$, to forms involving the scalar eigenfunctions $U_l, V_l, W_l$ and their derivatives. The result has already been derived in eqn [102] by a different route. Similar formulae are also to be found in Gilbert and Dziewonski (1975), Woodhouse and Girnius (1982), and Dziewonski and Woodhouse (1983b). Using eqn [117] in eqn [111], we obtain the following expression for a theoretical seismogram:

$$u(x, t) = \sum_{l m} \frac{1}{\omega_l^2} E_{l m} s^{(l m)}(x)(1 - e^{-\omega_l^2 t})$$  

The argument of the previous section shows that the correct form of this expression in the attenuating case is

$$u(x, t) = \sum_{l m} \text{Re} \left\{ \frac{1}{\omega_l^2} E_{l m} s^{(l m)}(x)(1 - e^{\text{km} t}) \right\}$$  

where $\omega_l$ is the complex frequency, and $s^{(l m)}$ is the complex eigenfunction, having normalization [99]. Additional terms need to be added to eqn [119] if it is desired to include relaxation effects. The actual seismogram is obtained by operating an ‘instrument vector’ in eqn [119].

This rather simple formula is the key ingredient of many seismological studies, as outlined in Section 1.03.1. Mode catalogs for PREM (Dziewonski and Anderson, 1981) exist up to 170 mHz (6 s period). Figure 2 shows an example of such a synthetic seismogram, calculated by summing all modes up to 6 s period. For comparison, the observed seismogram for the same earthquake and station is also shown. The direct P and S and surface reflected PP and SS body waves are visible in both the synthetic and data seismograms. At later times, the surface waves can be observed. Differences between the synthetic and data seismogram can be attributed to 3D structure, which is not included in the calculation.

### 1.03.7 The Normal Mode Spectrum

#### 1.03.7.1 Asymptotic Relations Relating Normal Modes, Body Waves, and Surface Waves

Here, we illustrate some of the properties of different multiplets in the normal mode spectrum. One way to gain insight into the physical character of modes is by relating them to traveling body waves and surface waves. The essential quantitative connection between modes and traveling waves is made by equating the horizontal wavelength (or wavenumber) of the mode with the corresponding horizontal wavelength (or wavenumber) of a traveling wave. For modes, this wavelength can be derived from the asymptotic properties of the spherical harmonics for large $l$. A point source at the pole $\theta = 0$ excites only the modes having low azimuthal order $|m| \leq 2$, as we have seen. For fixed $m$ and large $l$ we have (e.g., Abramowitz and Stegun, 1965):
Moving up in overtone number \( n \), the ranges of depth for which these inequalities are satisfied are indicated in two columns on the right side of each panel. The left column is for P-waves (relevant only for spheroidal multiplets) and the right column for S waves.

### 1.03.7.2 Differential Kernels for Spherically Symmetric Model Perturbations

Another way to gain understanding of the physical properties is through the use of differential kernels \( K(r) \). These are, in essence, the derivatives of the eigenfrequency of a given mode with respect to a structural change at any radius. This takes the form of an integral (cf. the chain rule applied to an infinite number of independent variables). Differential kernels can be defined, for example, corresponding to perturbations in shear modulus and bulk modulus, at fixed density, and these are related to the distribution of elastic shear energy and compressional energy with radius. Similarly, kernels can be defined corresponding to anisotropic perturbations (see Chapter 1.19). Theoretical formulae for such kernels can be found in, for example, Backus and Gilbert (1967), and for anisotropic elastic parameters \( A, C, L, N, F \) in Dziewonski and Anderson (1981). Such kernels are a special case, in which the perturbation is spherically symmetric, of the kernels that arise when a general aspherical perturbation of the model is considered. This will be further discussed in Section 1.03.8.2. Here, we shall take relative perturbations in (isotropic) P-velocity, \( v_P \), S-velocity, \( v_S \) and density \( \rho \) (at constant \( v_P \) and \( v_S \)) as the independent perturbations, and write

\[
\delta \omega_k = \int_0^a \left( K_P(r) \frac{\delta \rho(r)}{\rho(r)} + K_P(r) \frac{\delta v_P(r)}{v_P(r)} + K_S(r) \frac{\delta v_S(r)}{v_S(r)} \right) dr \tag{128}
\]

Our aim here is to use the kernels \( K_P(r) \), \( K_S(r) \) to give insight into the nature of the mode in terms of its traveling-wave content, and into how the P and S velocity distributions can be constrained by making observations of a given mode. The kernel \( K_P(r) \) (for constant \( v_P \) and \( v_S \)) gives information about how the mode probes the density structure, adding to information about \( v_P \) and \( v_S \) available from modes but also from travel times.

### 1.03.7.3 Tour of Frequency-Wavenumber Space

Here, we will discuss the characteristics of the spheroidal and toroidal modes. Figure 9 shows the degenerate eigenfrequencies of spheroidal modes as a function of angular order \( l \) for the spherical reference model PREM. Lines connect modes of the same overtone number \( n \), and define different branches of the dispersion curves. The fundamental mode branch \( (n = 0) \) contains the modes with the lowest frequency for each \( l \). Modes with \( n > 1 \) are called overtones. The overtone branches cross-cut each other leading to a range of different normal mode classes, which will be discussed in the succeeding text. Figure 10 shows the same for the toroidal modes, which are much simpler and do not have cross-cutting branches. We will make a tour of \( \omega - l \) space and use the eigenfunctions and differential kernels defined above to gain insight into the nature of the different types of mode.

Figures 6, 11–13 show eigenfunctions and differential kernels for a number of toroidal and spheroidal modes. By inspecting these diagrams, a few general observations can be made. Moving up along a mode branch (horizontal rows in Figures 11 and 12) will result in eigenfunctions and kernels which are more concentrated toward the surface. This reflects the fact that, for high \( l \), the modes may be interpreted as surface waves as is most obvious for the fundamental branches of the toroidal modes (Figure 11) and spheroidal modes (Figure 12). Moving up in overtone number \( n \), for constant \( l \) (vertical columns in Figures 11 and 12), leads to eigenfunctions that

\[ Y_l^m(\theta, \phi) \sim \frac{1}{\pi} \sqrt{\frac{2l+1}{4\pi}} \sin \theta \left( \frac{l+1}{2} \right)^{1/2} \left( \frac{1}{\sin \theta} \right)^{1/2} \left( \frac{2l+1}{4\pi} \right)^{1/4} e^{im\phi} \tag{120} \]

where \( \theta \) plays the role of epicentral distance. Thus, we can identify the horizontal wavenumber \( k \) (=2\pi/wavelength) to be

\[ k = \left( l + \frac{1}{2} \right)/a \tag{121} \]

The angular order \( l \), therefore, is a proxy for wavenumber \( k \) and dispersion diagrams such as those shown in Figures 9 and 10 can be interpreted, for large \( l \), in the same way as are dispersion relations \( \omega(k) \) for surface waves. In particular, we can define phase velocity

\[ c(\omega) = \frac{\omega}{k} \tag{122} \]

and group velocity

\[ U(\omega) = \frac{d\omega}{dk} \tag{123} \]

This defines the relationship between the \( \omega-l \) plane and the dispersion properties of Love and Rayleigh waves and their overtones, Love waves corresponding to toroidal modes and Rayleigh waves to spheroidal modes.

In the case of body waves we may, similarly, identify the horizontal wavenumber in terms of frequency and ray parameter \( p \) (Brune, 1964, 1966). From classical ray theory in the spherical Earth, the horizontal wavenumber at the Earth’s surface for a monochromatic signal traveling along a ray with given ray parameter \( p = d\tau/d\lambda \) is

\[ k = \frac{\omega p}{a} \tag{124} \]

Therefore, using eqn [121], we write

\[ p = \frac{l + \frac{1}{2}}{\omega} \tag{125} \]

Thus, a mode of angular order \( l \) and angular frequency \( \omega \) is associated with rays having the ray parameter given by eqn [125]. For toroidal modes these are S-rays, and for spheroidal modes they are both P- and S-rays. It is well known that rays exist only for ranges of depth for which

\[ \frac{r}{v_P(r)} \geq p \quad \text{for P-waves} \tag{126} \]

\[ \frac{r}{v_S(r)} \geq p \quad \text{for S-waves} \tag{127} \]

In the diagrams of Figures 6 and 11–13, the ranges of depth for which these inequalities are satisfied are indicated in two columns on the right side of each panel. The left column is for P-waves (relevant only for spheroidal multiplets) and the right column for S waves.
are more oscillatory and to sensitivity kernels that penetrate more deeply.

The wave motion of toroidal modes is purely horizontal and only has one eigenfunction \( W \) (see Figure 6). Thus, for spherical, nonrotating Earth models, these modes can only be observed on the horizontal components of a theoretical seismogram and not on the vertical component. Toroidal modes are sensitive only to perturbations in S-velocity and density and have no sensitivity to the fluid core (Figure 11). The differential kernels tell us how the frequency of the mode will change if we increase the spherical velocity or density at a certain depth. When inspecting Figure 11, we find that increasing the shear wave velocity at any depth in the mantle will always lead to an increase in toroidal mode frequency as the \( K_S(r) \) sensitivity kernel is always positive. For density, however, we find that it depends on the depth of the perturbation. For mode \( gT_2 \), for example, an increase in density in the upper mantle will lead to a decrease in frequency, while an increase in the lower mantle will increase the frequency. When we move from the fundamentals to the overtones, we find that the density kernel \( K_r(r) \) becomes oscillatory around zero (see, e.g., the \( n=5 \) overtones). These modes are almost insensitive to smooth variations in density, as the kernels will average to zero when integrated over depth. The sensitivity to density also becomes smaller for larger \( l \) along the same branch, which can clearly be seen when progressing from \( gT_2 \) to \( gT_{10} \) along the fundamental mode branch. This agrees well with the interpretation of shorter period toroidal modes in terms of Love waves, which are also dominated by sensitivity to shear wave velocity and have much smaller sensitivity to density.

Figure 12 shows the examples of sensitivity kernels for the spheroidal modes. The spheroidal modes involve wave motion in both horizontal and vertical directions characterized by eigenfunctions \( U(r) \) and \( V(r) \) (Figure 6), and so are sensitive to perturbations in density and to both \( v_P \) and \( v_S \). These modes are observed on both the horizontal and vertical components.
of theoretical seismograms for a spherical, nonrotating Earth model. Again, moving to the right along the fundamental mode branch in Figure 12 shows that sensitivities become progressively concentrated closer to the surface. Spheroidal modes correspond to Rayleigh waves and at higher $l$ the largest sensitivity is to shear wave velocity. This is similar to the toroidal modes, except that peak sensitivity is reached at subcrustal depths, making spheroidal modes less sensitive to large variations in shear velocity in the crust than is the case for toroidal modes.

The overtones sample different families of modes which relate to the branch crossings. Mode $S_{10}$ (Figure 12) corresponds to a Stoneley wave traveling along the inner core boundary. Stoneley waves are the analogue of a Rayleigh wave, but at a fluid–solid interface, rather than at a free surface. They exist both at the inner core boundary and at the core–mantle boundary.

These modes follow lines of increasing frequency $\omega$ and angular order $l$ in Figure 9, cross-cutting the branches with constant overtone number $n$. Mode $S_{10}$ (Figure 12) is a mixture of mantle mode and a Stoneley wave at the core–mantle boundary. The other modes in Figure 12 show a behavior similar to the toroidal modes. Notice that for $S_{20}$ the $v_p$ sensitivity decays below the P-wave ray-theoretic turning point (the point at which the shading terminates in the left vertical stripe at the right of the plot), and the $v_s$ sensitivity decays below the S-wave turning point. The fact that, for a given ray parameter, S-waves turn at greater depth than P-waves means that in modeling there is some potential for shallow P-velocity structure to trade off with deep S-velocity structure.

Figure 13 shows another family of spheroidal modes, which are characterized by low angular order $l$ and high overtone number $n$. These modes are the PKIKP equivalent modes which

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**Figure 12** Sensitivity kernels of selected spheroidal modes. $K_S(r)$ dashed, $K_P(r)$ solid, and $K_r(r)$ dot-dashed.
have strong sensitivity to core structure. Modes $3S_2$ and $10S_2$ have strong sensitivity to inner core $u_s$, while $13S_2$ is more strongly sensitive to $u_p$. These modes display anomalous splitting which has been attributed to inner core anisotropy (Woodhouse et al., 1986) and its hemispherical variation (Deuss et al., 2010).

Finally, Figure 14 shows sensitivity kernels for radial modes, which are yet another subgroup of the spheroidal modes with angular order $l = 0$. The radial modes are sensitive to both $u_s$ and $u_p$ and also to density. However, the sensitivities to $u_s$ and density oscillate around zero, while only for $u_p$ the sensitivity is always positive. This means that radial modes are very useful for obtaining constraints on the one-dimensional compressional structure of the Earth.

1.03.8.1 Theoretical Seismograms

The problem of calculating theoretical seismograms and spectra for three-dimensional models is a challenging one. A theoretically straightforward formalism exists for such calculations, based on expanding the wave field in terms of a complete set of (vector) functions. The equations that arise from requiring that the equations of motion be satisfied can be regarded as exact matrix equations. However, they are of infinite dimension, and even when truncated to give a practicable method of solution, the calculations require the manipulation and diagonalization of extremely large matrices. This is already problematic in the forward problem, but becomes even more so for the inverse problem, requiring calculations which are equivalent to thousands of forward problems. If the expansion is carried out in terms of the eigenfunctions of a spherical model (that is close to reality), the process is simplified, as the off-diagonal terms in the resulting matrices are small,
allowing some approximate schemes that are much less cumbersome to be developed.

Let us take the equation of motion in operator form (given in eqn \([36]\)) as our starting point in considering the form that this theory takes. In the frequency domain:

\[
(\mathbf{H} - \rho_0^2) \mathbf{u} = \mathbf{f} \tag{129}
\]

The Earth will be considered to be in a state of steady rotation with angular velocity \(\Omega\) about its center of mass, so that the expression for \(\mathbf{H}\) acquires additional terms representing the Coriolis force and the centrifugal potential (Dahlen, 1968):

\[
(\mathbf{H}(\mathbf{u})) = \rho \left( \phi^0 + \left[ \phi^0 - \frac{1}{2} \Omega^2 \mathbf{Q}^0 \cdot \mathbf{x}_0 \right] \right) \mathbf{u} + 3 \mathbf{F}_0 \mathbf{Q}^0 \mathbf{u} \]

\[
- (A_{ijkl} \mathbf{u})_{klij} \tag{130}
\]

where, as previously, \(\phi^0\) is regarded as a functional of \(\mathbf{u}\), by virtue of Poisson's equation \([23]\). The terms on the right-hand side of eqn \([129]\) involving \(\Omega^2 \mathbf{Q}^0 \cdot \mathbf{x}_0 \) relate to the centrifugal potential and the term involving \(3 \mathbf{F}_0 \mathbf{Q}^0 \) is the Coriolis force contribution. Let \(\mathbf{s}^{(lm)}\) represent the eigenfunctions of a spherical, nonrotating, nonattenuating reference model, with eigenfrequencies \(\omega_{0i}\) so that

\[
\mathbf{H}_{0\mathbf{s}}^{(lm)} = \rho_0 \omega_{0i}^2 \mathbf{s}^{(lm)} \tag{131}
\]

satisfying the orthogonality relation

\[
\int_V \rho_0 \mathbf{s}^{(lm)*} \cdot \mathbf{s}^{(lm')} \ dV = \delta_{\ell\ell'} \delta_{mm'} \tag{132}
\]

where \(\mathbf{H}_0\) and \(\rho_0\) are for the spherical, nonrotating, nonattenuating reference model. Now let us seek a solution of eqn \([129]\) in terms of an expansion

\[
\mathbf{u} (\mathbf{x}, \omega) = \sum_{lm} a_{lm} \mathbf{s}^{(lm)} (\mathbf{x}) \tag{133}
\]

with coefficients \(a_{lm}\) to be found. Substituting the expansion for \(\mathbf{u} (\mathbf{x}, \omega)\) into eqn \([129]\), and then taking the dot product with \(\mathbf{s}^{(lm)*}\) and integrating, we find

\[
\sum_{lm} \left[ (k'm') \mathbf{H}_1 - \rho_1 \omega^2 \right] \mathbf{k} | \mathbf{k} \rangle \langle \mathbf{l} | m \rangle - (\omega^2 - \omega_0^2) \delta_{kk'} \delta_{mm'} \right] a_{lm} = (k'm') \mathbf{f} \tag{134}
\]

where \(\mathbf{H}_1 = \mathbf{H} - \rho_0 \mathbf{u}\), \(\rho_1 = \rho - \rho_0\) and where we have introduced the notations \((k'm') = \int_V \mathbf{s}^{(lm)*} \cdot \mathbf{f} \ dV\), 

\[
\mathbf{H}_1 - \rho_1 \omega^2 \mathbf{k} | \mathbf{l} | m \rangle = \int_V \mathbf{s}^{(lm)*} (k'm') \mathbf{H}_1 - \rho_1 \omega^2 \mathbf{k} | \mathbf{l} | m \rangle \]

Equation \([134]\) can be regarded as a matrix equation, albeit of infinite dimension, in which rows and columns of the matrix on the left side are labeled by \((k', m'), (k, m)\), respectively and in which the rows of the column on the right side are labeled by \((k', m')\). We can write

\[
\mathbf{C}(\omega) \mathbf{a} = \frac{1}{i\omega} \mathbf{E} \tag{135}
\]

where \(\mathbf{C}(\omega)\) is the matrix having eigenvalues \((k'm' | \mathbf{C}(\omega) | \mathbf{k} \rangle \langle \mathbf{l} | m \rangle = (k'm' | \mathbf{H}_1 - \rho_1 \omega^2 \mathbf{k} | \mathbf{l} | m \rangle - (\omega^2 - \omega_0^2) \delta_{kk'} \delta_{mm'} \)

and where \(\mathbf{E}(\omega)\) is the column vector having elements \(k'm' \mathbf{f}\), the factor \(1/i\omega\) being inserted to reflect the assumed step-function.

Figure 15: Schematic depiction of the different blocks in the matrix \(\mathbf{C}(\omega)\) for the coupling of modes. Self-coupling blocks (dark gray) on the diagonal are of size \((2i + 1) \times (2i + 1)\) and one exists for each mode \(n_S^0\). Cross-coupling blocks (light gray) are of size \((2j + 1) \times (2j + 1)\) and exists for each pair of modes \(n_S^0\) and \(n_S^j\).
frequencies \( \omega = \omega_0 \), say, for which there exists a nontrivial solution to the homogeneous problem
\[
C(\omega) r_k = 0 \quad [137]
\]
This is an eigenvalue problem for \( \omega_0 \) and the corresponding right-eigenvector \( r_k \).

The eigenvalue problem is of a nonstandard form, since the dependence of \( C \) on the eigenvalue parameter \( \lambda = \omega^2 \), is not of the usual form \( C_0 - \lambda I \). This is because, (i) perturbations in density \( \rho_1 \) introduce a more general dependence on \( \omega^2 \), (ii) by virtue of attenuation, the perturbations in the elastic parameters entering into \( H_1 \) are dependent upon \( \omega_0 \), (iii) the terms arising from rotation depend upon \( \omega_0 \) rather than on \( \omega^2 \). Thus, the equation for \( C(\omega) \) is of the form
\[
C(\omega) = V(\omega) + i\omega W - \omega^2 T - (\omega^2 - \omega_0^2) I \quad [138]
\]
where \( V(\omega) \) is the potential energy operator which incorporates the contributions from perturbations in the elastic parameters \( A_{ijkl} \) and the centrifugal potential, \( W \) is the Coriolis rotation operator, \( T \) is the kinetic energy operator which contains the perturbations to density \( \rho_1 \), and \( I \) is the identity matrix. (For a spherical, nonrotating model, \( V = W = T = 0 \) and we find that the eigenvalues are given by \( \omega_0 = \omega_0 \), i.e., the eigenvalues of the spherical, nonrotating reference model.)

The form of the solution as a sum over residues arising from singularities at \( \omega = \omega_0 \) can be obtained by replacing the inverse Fourier transform by a summation over singularities \( h \), and within the integrand carrying out the replacement (Deuss and Woodhouse, 2004, cf. eqn [98]):
\[
C(\omega)^{-1} \rightarrow \sum_{l_h} \frac{r_k l_h}{l_h \partial_{\omega_h} C(\omega_h) r_k} \quad [139]
\]
where \( l_h \) is the left-eigenvector (a row rather than a column), the solution of
\[
I_h C(\omega_h) = 0 \quad [140]
\]
We are assuming here that eqns [137] and [140] determine the right and left eigenvectors \( r_k \), \( l_h \) uniquely, up to multiplying factors, such factors being immaterial for the evaluation of the residue contribution [139]. Thus, from eqn [133] the solution in the time domain can be written
\[
u(x, t) = 2\Re \sum_{km} \frac{1}{l_h} \frac{l_h E}{l_h \partial_{\omega_h} C(\omega_h) r_k} (km/r_k) s^{(km)}(x) e^{i\omega_0 t} \quad [141]
\]
where the notation \( (km/r_k) \) represents individual elements of the column \( r_k \). As previously, the contribution from singularities in the left half of the complex \( \omega \)-plane is incorporated by taking twice the real part, the summation in eqn [141] being taken only for \( \omega_0 \) in the right half-plane.

In order to make use of this theory it is necessary to obtain expressions for the matrix elements \( \langle k' m' | C(\omega) | km \rangle \) in terms of the perturbations in elastic parameters, density, etc. and deviations of surfaces of discontinuity from the spherically symmetric reference model. Fairly complete forms for these are given by Woodhouse (1980a), omitting terms in anisotropic parameters and in initial stress. For anisotropic perturbations, see Chapter 1.19. The basic method is to expand the perturbations in spherical harmonics \( P' \), \( m' \), and then to evaluate the integrals of triply of spherical harmonics using the formula, derivable from eqns [47] and [52],
\[
\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \left( Y_{l'm'}^* \right)^* Y_{l''m''} Y_{l''m''} \sin \theta d\theta d\phi = (-1)^{l''-m''} \begin{pmatrix} l & l & 1 \\ -N' & N'' & m \\ m' & m'' & m \end{pmatrix} (-N' = N'' + N) \quad [142]
\]
The resulting form for the matrix elements can then be written in the form
\[
\langle k' m' | C(\omega) | km \rangle = \sum_{l''} (-1)^{l''-m''} \begin{pmatrix} l & l & 1 \\ -m' & m'' & m \end{pmatrix} \langle k' | C(l''m'') \rangle (\omega) \langle | k \rangle \quad [143]
\]
where the so-called reduced matrix element appearing in the right side, itself defined by this equation, is independent of \( m \) and \( m' \). This particular form for the dependence of the matrix elements on \( m \) and \( m' \) is a consequence of the Wigner–Eckart theorem (see Edmonds, 1960). The expressions for the reduced matrix elements take the form of radial integrals involving pairs of scalar eigenfunctions, for multiplets \( l' \) and \( k \), and on the \( (l''m'') \) component of the spherical harmonic expansion of heterogeneity, together with terms evaluated at boundaries corresponding to the \( (l''m'') \) components of the deflections of the boundaries.

### 1.03.8.2 Splitting of Isolated Modes: The Splitting Function

The theory above is fairly complete for the oscillations of a general Earth model. Apart from the treatment of spherical boundary perturbations, which involves a linearization of the boundary conditions, it is in principal an exact theory (Woodhouse, 1983), provided that coupling between all multipoles is taken into account – that is, provided that the eigenvalue problem for \( \omega_0 \) includes all the blocks (i.e., all modes) of the full matrix \( C(\omega) \). Of course the theory cannot be applied in its full form, owing to the need to manipulate infinite-dimensional matrices, and so a number of approximate schemes have been developed. The simplest, and up to now most widely applied method is to reduce the eigenvalue problem for \( \omega_0 \) to that for a single diagonal block or mode – the so-called self-coupling approximation. In this case, we focus on a single multiplet, \( k \), and reduce \( C(\omega) \) to the \((2l+1) \times (2l+1)\) block matrix corresponding to multiplet \( k \), see Figure 15. This is a justifiable approximation for the calculation of \( u(x, \omega) \) for frequencies near \( \omega_0 \) if the mode can be considered isolated, which is to say that there are no other modes nearby having similar frequency and significant coupling terms. A precise statement of the conditions that need to be satisfied for a mode to be considered isolated has not, to our knowledge, been worked out but, roughly speaking, it is necessary for the ratio \( \langle k' m' | C(\omega) | km \rangle / (\omega_0^2 - \omega_0^2) \) for all other multipoles \( k' \) to be small, for \( \omega \) near the frequency \( \omega_0 \) of the target multiplet.

In the self-coupling approximation the dependence of \( C(\omega) \) can be linearized for frequencies near \( \omega_0 \), \( C(\omega) \approx C(\omega_0) + C'(\omega_0) \delta \omega \), and a \((2l+1) \times (2l+1)\) matrix eigenvalue problem is obtained for \( \delta \omega \):
having eigenvalues \( \delta_{\omega_{0k}} \), say. This can also be written, to zeroth order

\[
\mathbf{H}^{(k)} \delta_{\omega_{0k}} = 0
\]

[144]

where the \((2l+1) \times (2l+1)\) matrix \( \mathbf{H}^{(k)} \), called the splitting matrix of the target multiplet, \( k \), has elements \( (\mathbf{H}^{(k)})_{\mathbf{m}\mathbf{n}} = (\mathbf{H}^{(k)})_{\mathbf{m}\mathbf{n}} / \Omega_{\mathbf{m}\mathbf{n}} \). This has now become a standard eigenvalue problem, involving diagonalization of the matrix \( \mathbf{H} = \delta_{\omega_{0k}} \), which can easily be performed. The matrix \( \mathbf{H}^{(k)} \) is given by

\[
\mathbf{H}^{(k)} = \frac{1}{\Omega_{\mathbf{m}\mathbf{n}}} \mathbf{H}_{\mathbf{m}\mathbf{n}} = \frac{1}{\Omega_{\mathbf{m}\mathbf{n}}} (\mathbf{V}(\omega_{0k}) + i \omega_{0k} \mathbf{W} - \omega_{0k}^2 \mathbf{I})
\]

[146]

The contribution to the right side of eqn [141] can be written (Woodhouse and Gurniis, 1982)

\[
u_k(x, t) = \text{Re} \sum_{\mathbf{m}\mathbf{n}} \left( \frac{1}{\Omega_{\mathbf{m}\mathbf{n}}} \left( \exp \left( i \mathbf{H}^{(k)} t \right) \right)_{\mathbf{m}\mathbf{n}} E_{\mathbf{m}\mathbf{n}} \right) \mathbf{s}^{(\mathbf{m}\mathbf{n})}(x) e^{i\omega_{0k} t}
\]

[147]

the matrix exponential arising from the identity

\[
\sum_{l=0}^{2l+1} \frac{t^l}{l!} \frac{\delta_{\omega_{0k}}}{\delta_{\omega_{0k}}} \exp(i\omega_{0k} t) = \exp(\mathbf{H} t)
\]

[148]

where \( \delta_{\omega_{0k}} \) are the eigenvalues of \( \mathbf{H} \) and \( t = \frac{\omega_{0k}}{\mu} \) is right and left eigenvectors; the division by \( \frac{\omega_{0k}}{\mu} \) ensures that the eigenvectors are properly normalized. Equation [147], which is directly comparable to the result for the spherical reference model in eqn [119] (The static, time-independent terms are not included, as we are considering an approximation valid only in the spectral neighborhood of \( \omega_{0k} \), the frequency of the target multiplet), has the simple interpretation that at time \( t = 0 \) the modes are excited as they would be in the reference model, as the matrix exponential is initially equal to the unit matrix. With time, the effective excitation \( \exp(\mathbf{H}^{(k)} t) E_{\mathbf{m}\mathbf{n}} \) evolves on a slow time scale characterized by the incremental eigenfrequencies \( \delta_{\omega_{0k}} \), the eigenvalues of \( \mathbf{H}^{(k)} \). In the frequency domain, this leads to splitting of the degenerate eigenfrequency \( \omega_{0k} \) into \( 2l+1 \) singlets — hence the name splitting matrix for \( \mathbf{H}^{(k)} \).

It is straightforward to set up the inverse problem of estimating the splitting matrix for isolated multiplets using data spectra for many events. This is simplified by recognizing that the \((2l+1) \times (2l+1)\) matrix \( \mathbf{H}^{(k)} \) is equivalent to a certain function on the sphere, known as the splitting function (Woodhouse and Gurniis, 1985). It can be shown that for scalar perturbations from the reference model, such as \( \rho_{l\mu}, \kappa_{l\mu}, \mu_{l\mu} \), \( \mathbf{H}^{(k)} \) is expressible in terms of coefficients \( c_{l''m''}^{(k)} \) which represent the spherical harmonic expansion coefficients of even degree \( l'' \), up to finite spherical harmonic degree \( l'' \leq 2l \) by the expression

\[
(\mathbf{m'} \mathbf{n'} | \mathbf{H}^{(k)} | \mathbf{m} \mathbf{n}) = \Omega_{\mathbf{m} \mathbf{n}} c_{l''m''}^{(k)} + \Omega_{\mathbf{m} \mathbf{n}} \sum_{l'' \leq l} \sum_{m''=-l''}^{l''} (-1)^{m''} \left( \frac{2l'' + 1}{4\pi} \right)^{1/2} \times (2l+1) \left( \frac{1}{0} \right) \left( \frac{1}{0} \right) \left( \frac{-1}{m''} \right) (l'' m'' l m') c_{l''m''}^{(k)}
\]

[149]

where the first term is the effect of Coriolis forces (Dahlen, 1968), \( \beta \) being the (known) rotational splitting parameter for the multiplet. Thus, the inverse problem for \( \mathbf{H}^{(k)} \) is equivalent to the estimation of \( c_{l''m''}^{(k)} \). The function on the sphere

\[
\eta(\theta, \phi) = \sum_{l'' = 0}^{l} \sum_{m'' = -l''}^{l''} c_{l''m''}^{(k)} \exp(i l'' m'' \theta) Y_{l''m''}^*(\theta, \phi)
\]

[150]

can be interpreted, at least for high-\( l \) modes, as the even degree expansion of \( \delta_{\omega_{0k}} \), in which \( \delta_{\omega_{0k}} \) is the eigenfrequency that a spherically symmetric model would possess if its radial structure were the same as the structure beneath the point \( (\theta, \phi) \) (Jordan, 1978). Only even degrees are present by virtue of the fact that the first 3-\( j \) symbol in eqn [149] vanishes for odd values of \( l'' \). Thus, self-coupling is only sensitive to even degree structure, that is, the average between pairs of antipodal points. Clearly, this is not the case for traveling waves and represents a shortcoming of splitting theory.

The splitting function leads to a two-stage inversion for three-dimensional structure in which stage 1 is to find the structure coefficients \( c_{l''m''}^{(k)} \) that bring data and theoretical spectra into agreement, using as many events and stations as are available. Stage 2 is to determine the structural perturbations to internal heterogeneity of the Earth of degree \( l'' \) and \( m'' \) needed to match the inferred values of \( c_{l''m''}^{(k)} \). Stage 1 of the procedure is nonlinear, owing to the fact that the relation between the synthetics \( u(x, t) \) and structure coefficients \( c_{l''m''}^{(k)} \) involves the exponential \( \exp(\mathbf{H}^{(k)} t) \). Stage 2, on the other hand, is linear; \( c_{l''m''}^{(k)} \) is related to three-dimensional structural perturbations by integrals involving known differential kernels, that is

\[
\eta(\theta, \phi) = \int_0^\infty \left( K_{l''}^\rho(r) \delta\rho^{m''}(r) \rho(r) + K_{l''}^\kappa(r) \delta\kappa^{m''}(r) \kappa(r) + K_{l''}^\mu(r) \delta\mu^{m''}(r) \mu(r) \right) dr + \sum_d \delta h_{l''m''}^d H_{l''m''}^{(k)}
\]

[151]

where \( K_{l''}^\rho(r), K_{l''}^\kappa(r), K_{l''}^\mu(r) \) are known kernels for density, compressional velocity, shear wave velocity perturbation and \( H_{l''m''}^{(k)} \) for discontinuity perturbations such as crustal thickness and CMB topography. Practical formulas for calculating these kernels are given in Woodhouse (1980a). This is a similar procedure to that commonly employed in surface wave studies, in which one first determines two-dimensional maps of phase velocity, over a range of frequencies, and then uses these to infer the three-dimensional structure perturbations needed to explain the inferred phase velocity maps. The spectral fitting approach using splitting function coefficients \( c_{l''m''}^{(k)} \) has been widely applied (e.g., Deuss et al., 2013; Giardini et al., 1987, 1988; He and Tromp, 1996; Masters et al., 2000; Resovsky and Ritzwoller, 1995, 1998; Ritzwoller et al., 1988; Romanowicz and Breger, 2000).

Figure 16 shows an example of data and theoretical spectra using the self-coupling approach, and including splitting function coefficients estimated from a large collection of data. In the left panel only the splitting effects of rotation and ellipticity are taken into account, whereas in the right panel the estimated splitting function has been used to calculate the synthetic spectra. The effects of ellipticity and rotation can be calculated for the spherical reference model and do not require knowledge of the three-dimensional structure. While rotation and ellipticity lead to some splitting of the singlets (Figure 16(a)), further splitting is present which is attributed
to three-dimensional structure and can be accounted for by using the splitting function coefficients in the calculation of the theoretical spectra (Figure 16(b)). The distribution of singlets and their excitations is known only by virtue of the inversion itself. There is no possibility here of resolving the singlets in individual spectra, but by modeling a large collection of spectra, for many events and stations, the underlying singlet distribution is unmasked. This example illustrates the fact that there are large differences between data and synthetics prior to estimating the splitting function coefficients, indicating that long-period spectra represent a rich source of information about the Earth’s three-dimensional structure.

Figure 17 shows splitting function maps (i.e., using eqn [150]) which are measured by nonlinear inversion of large numbers of spectra such as shown in Figure 16. This is stage one of the inversion procedure. The top panel (Figure 17(a)) shows mode $s_{14}$ which is sensitive to shear wave velocity in most of the mantle and peaks in compressional velocity sensitive at the CMB; this mode is almost a Stoneley mode. The observed splitting function for $s_{14}$ shows higher frequencies in the ‘ring around the Pacific,’ which is also predicted for the mantle tomographic model S40RTS (Ritsema et al., 2011). These higher frequencies are due to higher velocity anomalies which are found in tomographic models in the regions of the subduction zones around the Pacific. Mode $s_{22}$ (Figure 17(b)) is mainly sensitive to upper mantle shear wave velocity, and is very well predicted by mantle shear wave velocity model S40RTS. Mode $s_{32}$ (Figure 17(c)) is predominantly sensitive to compressional velocity, and is less well predicted by shear wave velocity model S40RTS. Finally, mode $s_{32}$ (Figure 17(d)) is sensitive to the inner core, and displays strong zonal splitting (i.e., higher frequency anomalies near the poles and low frequency anomaly along the equator). This signature is not predicted by mantle models alone and requires inner core anisotropy (e.g., Woodhouse et al., 1986).

The splitting function approach has been used in inversions for tomographic velocity models (Ishii and Tromp, 1999; Li et al., 1991; Resovsky and Ritzwoller, 1999). Some recent tomographic shear wave velocity models, such as S20RTS and S40RTS (Ritsema et al., 1999, 2011), make use of splitting functions in addition to body wave, surface wave, and overtone data to provide improved constraints on the low degree structure. Splitting functions have also been used in the discovery of inner core anisotropy (Woodhouse et al., 1986), its hemispherical variation (Deuss et al., 2010) and have provided constraints on the possible rotation of the inner core (Laske and Masters, 1999; Sharrock and Woodhouse, 1998).

1.03.8.3 Normal Mode Coupling

The fact that in the self-coupling approximation seismic spectra depend only upon the even spherical harmonic degrees of heterogeneity points to a shortcoming of the theory. Since spherical harmonics of even degree are symmetric under point reflection in the center of the Earth, self-coupling theory predicts that the seismic spectra depend only upon the average structure between pairs of antipodal points. Thus, the interaction, or coupling, of modes must be a key effect for understanding wave phenomena that do not have this symmetry property. Figure 18 shows the combined dispersion diagrams for spheroidal and toroidal modes at low frequencies ($f \leq 3$ MHz), an expanded version of the lower left corner of the dispersion diagrams in Figures 9 and 10. Many modes of different branches or even type (e.g., spheroidal or toroidal) have very similar frequencies, which leads to ‘resonance’ between the singlets of the different modal multiplets; we call...
An unusual degree

Spheroidal and toroidal modes

Figure 18  Eigenfrequencies of spheroidal (black circles) and toroidal multiplets (white circles) below 3 mHz for the preliminary reference model (PREM, Dziewonski and Anderson, 1981). A multiplet with angular order $l$ consists of $2l+1$ singlets with azimuthal order $m = -l, -l+1, \cdots, l-1, l$. The branches are labeled by their overtone number $n$ (left spheroidal, right toroidal), the fundamental mode branch is $n = 0$.

Figure 17  Observed splitting function maps and predictions for mantle model S20RTS. $N_s$ gives the total number of spectra used for the splitting function measurement. The misfit is calculated between the observed spectra and theoretical spectra and is found to be much smaller using the observed splitting functions instead of the predictions for the S40RTS mantle shear wave velocity model (Ritsema et al., 2011) and crustal model CRUST5.1. The left panels show the sensitive kernels for each mode, where red is $u_p$, solid black is $u_s$, and dashed is density. Reproduced from Deuss A, Ritsema J, and van Heijst HJ (2013) A new catalogue of normal-mode splitting function measurements up to 10 mHz. Geophysical Journal International 193: 920–937.
this effect cross-coupling in seismology. Coupling between modes allows for sensitivity to both even and odd degree structure. Two spheroidal modes $\psi_{l_p} \omega_{l_p}$ will be sensitive to structure of degree $|l-p| \leq l' \leq |l+p|$, and requiring that $l+l' \neq p'$ is even. So, if $l+p'$ is odd, there will be sensitivity to odd-degree structure. And if $l+l'$ is even (and the special case of self-coupling when $l+l'$), then there will only be sensitivity to even degree structure. The same holds for pairs of toroidal modes. Pairs of spheroidal and toroidal modes also couple due to heterogeneity of degree $|l-p|+1 \leq l' \leq |l+p|$, and requiring that $l+l' + p'$ is odd. Additional coupling exists due to the Coriolis force and ellipticity, which both couple pairs of spheroidal and toroidal modes which differ by one in angular order, that is, $\psi_{l_p} \omega_{l_p}$. Ellipticity also couples same type modes which differ in angular order by zero or two. Ellipticity coupling can also be seen as special case of $l' = 2$ structure.

The theory can be straightforwardly extended to include the coupling of groups of modes. The resulting method is known as quasi degenerate perturbation theory (Dahlen, 1969; Luh, 1973, 1974; Woodhouse, 1980a), or group coupling. A small group of multiples $\{k_i, k_2, k_3, \ldots\}$, close in frequency, is selected, and the eigenvalue problem is reduced to that for the matrix obtained from $C(\omega)$ by selecting only the blocks corresponding to the chosen multiplets. This problem can then be linearized in $\delta\omega$, relative to a fiducial frequency in the chosen band, in much the same way as in the case of self-coupling, outlined above, the resulting matrix eigenvalue problem being of dimension $(2l_1+1)(2l_2+1)\ldots$ (Figure 15). The selected group of modes is said to form a super-multiplet. Resovsky and Ritzwoller (1995) have generalized the notion of the splitting function and structure coefficients to include coupling between pairs of multiplets, so that $c_{\psi_{l_p}\omega_{l_p}}$ becomes $c_{\psi_{l_p}\omega_{l_p}}$. And estimates have been made of such coefficients from seismic spectra (see, e.g., Deuss et al., 2013; Resovsky and Ritzwoller, 1998). While these have lead to odd-degree splitting function coefficients, they have not yet been included in tomographic inversions. Of particular interest, though, is the observation of hemispherical variations in inner core anisotropy (a type of odd-degree structure) which has been made using cross-coupling between pairs of inner core sensitive and inner core confined modes (Deuss et al., 2010).

The self-coupling and group coupling techniques depend upon the assumption that further cross-coupling is not needed to approximate the complete solution, which as we have shown, includes coupling between all multiplets. Of course full coupling calculations cannot be done for a truly infinite set of modes, but it is feasible at low frequencies to include coupling between all multiplets below a specified frequency. We shall call this full coupling. Deuss and Woodhouse (2001) have compared the different approximations used in computing normal mode spectra, and have found that self-coupling and group coupling can be a poor approximation to full coupling, indicating that a more complete version of the theory will need to be used in the future as it is desired to constrain the three-dimensional distribution of parameters, such as density, attenuation, and mantle anisotropy, on which the spectra depend more subtly.

Figure 19 shows a comparison between data and spectra calculated using self-coupling with those resulting from a full coupling calculation in which the coupling of all 140 modes up to 3 mHz has been included (see Deuss and Woodhouse, 2001, for details of the calculation). The spheroidal modes are clearly seen, and there is also signal for toroidal mode $\alpha T_{10}$ on the vertical component, which is due to Coriolis coupling. In the self-coupling calculation $\alpha T_{10}$ is not seen, but it appears in the full coupling calculation which includes Coriolis coupling between spheroidal mode $\psi S_2$ and toroidal mode $\alpha T_{10}$. There is reasonable agreement between the data and full coupling synthetics, but the differences between data and synthetics are comparable to the difference between the self-coupling and full coupling synthetics. It may be expected that group coupling would be justified, and that coupling among wide bands of modes can be ignored. However, coupling on groups still shows significant differences compared to full-coupling (see Figure 20).

Figure 19 Data and synthetics for modes $\psi S_2, \psi T_{10}, \psi S_0, \psi T_{10},$ and $2S_6$ at station PAB for an earthquake in Bolivia. The time window is 5–45 h. The differences between full coupling and self-coupling are similar to the differences between the data and full coupling and are of the same order as the differences that one would attempt to model. This indicates that full coupling is essential in future attempts to model inhomogeneous structure. Reproduced from Deuss A and Woodhouse JH (2001) Theoretical free oscillation spectra: The importance of wide band coupling. Geophysical Journal International 146: 833–842.
In principle, normal mode spectra can be inverted directly to derive tomographic models, avoiding the intermediate step of estimating the splitting function coefficients (Durek and Romanowicz, 1999; Hara and Geller, 2000; Kuo and Romanowicz, 2002; Li et al., 1991). This leads to a one-step inversion procedure in which model parameter adjustments that enable the data and theoretical spectra to be brought into agreement are sought directly. This scheme has the advantage that the full coupling approach can be used for the solution of the forward problem and for calculations of the derivatives needed for formulating the inverse problem. Of course, it has the disadvantage that a nonlinear inverse problem needs to be solved within a large model space, rather than being able to restrict the nonlinear stage of inversion to the much smaller number of parameters represented by splitting functions. The calculations are also much more burdensome in terms of computer time and memory. This means that the splitting function technique is still largely the preferred method in spectral fitting studies; however, to investigate the large regions of the spectrum where wide-band coupling is expected to be significant, a final stage inversion involving full coupling will be needed.

Significant theoretical work has been directed towards developing methods able to give accurate splitting and coupling results using a practicable amount of computer time and memory (e.g., Lognonne, 1991; Lognonne and Romanowicz, 1990; Park, 1990). Deuss and Woodhouse (2004) have developed a technique for solving the full coupling generalized eigenvalue problem ([137], [140]) by an iterative technique, not requiring the eigenvalue decomposition of very large matrices, which is well suited to the accurate modeling of small spectral segments. The first iteration of this technique is similar to the subspace projection method of Park (1990), which similarly aims to approximate full coupling effects while avoiding the need to find the eigenvectors and eigenvalues of very large matrices. Al-Attar et al. (2012) developed an iterative version of the direct solution method which also does not require the decomposition of very large matrices and has been shown to be very effective in coupling large groups of normal modes.

1.03.8.4 Centroid-Moment Tensor Method

A very useful application of normal mode theory is the centroid moment tensor method, which uses waveform data to derive both the source mechanism of an earthquake and also the hypocentral coordinates of the ‘best point source’ at a given frequency. This method was introduced by Dziewonski et al. (1981) and applied on a larger scale by Dziewonski and Woodhouse (1983a). Consider the moment tensor as defined in eqn [31]. The moment tensor approach depends on finding the best point source, which is called the centroid location and the corresponding centroid time for which the first spatial and temporal moments are minimized in the least squares sense. In this case, we can rewrite the synthetic seismogram [119] as

\[ u_k(x, t) = \sum_{i=1}^{6} \psi_{ik}(x, x_s, t) f_i(t) \]  \[ [152] \]

where \( f_i(t) \) represent the six independent components of the moment rate tensor, that is, \( f_1 = \mathbf{M}_{rx}, f_2 = \mathbf{M}_{ry}, f_3 = \mathbf{M}_{rz}, f_4 = \mathbf{M}_{ox}, f_5 = \mathbf{M}_{oy}, f_6 = \mathbf{M}_{oz}. \) The receiver is at position \( x \) and the source at \( x_s = (\theta_s, \phi_s, r_s). \) The excitation kernels \( \psi_{ik} \) depend on the properties of the Earth in terms of the eigenfunctions and eigenvalues of the normal modes and represent synthetic seismograms corresponding to the individual components of the moment tensor of unit amplitude. If the earthquake location is fixed, then eqn [152] becomes linear in the six moment tensor components \( f_i. \) Hence, the starting solution is obtained by solving eqn [152], for a starting source position \( x_s \) determined from, for example, first P and S arrivals. Having found the initial estimate of the seismic moment tensor \( f_i^{(0)} \), the process is then iterated using

\[ u_k - u_k^{(0)} = b_k \delta \tau + c_k \delta \phi_1 + d_k \delta \phi_2 + e_k \delta \phi_3 + \sum_{i=1}^{6} \psi_{ik}^{(0)} \delta f_i \]  \[ [153] \]

where \( b_k, c_k, d_k, e_k \) are the partial derivatives with respect to the hypocentral coordinates. \( u_k^{(0)} \) is the theoretical displacement calculated for the starting source coordinates. The theory so far is standard, but the special point is that the kernels will be

Figure 20 Synthetics for modes \( 2S_{13}, 0T_{21}, 0S_{20}, 0S_1, \) and \( 2T_8 \) for stations PAS and BJT of the Bolivia event of 9 June 1994. The time window is 5–80 h. All modes in the frequency interval shown were allowed to couple for the group coupling calculations; the full coupling calculation includes all modes up to 3 mHz. Reproduced from Deuss A and Woodhouse JH (2001) Theoretical free oscillation spectra: the importance of wide band coupling. Geophysical Journal International 146: 833–842.
calculated using normal mode summation, see the appendix of Dziewonski et al. (1981) for specific formula. There are ten unknowns to be determined, that is, the centroid time, the three centroid location parameters, and the six moment tensor components. The process keeps on being repeated until convergence is achieved.

Dziewonski et al. (1981) initially developed the method for $45 < T < 60$ s long-period body waves. The method was then extended by Dziewonski and Woodhouse (1983a) to include long-period surface waves with $135 < T < 150$ s. It was modified by Woodhouse and Dziewonski (1984) to include simultaneous inversion for three-dimensional Earth models by using the great circle approximation (see Section 1.03.8.5). The method was then enhanced again in 2004 to incorporate intermediate period surface waves (Ekstrom et al., 2012). The CMT mechanisms are now determined on a routine scale for all earthquakes with moments as small as $M_w = 5.0$, with results published on the website: http://www.globalcmt.org.

1.03.8.5 Great Circle Approximation

The development of practical and less time-consuming methods has so far been most successful in the long-period waveform and surface wave approach. Woodhouse and Dziewonski (1984) developed the elegant path-average great circle approximation using phase integrals along the minor and major arcs. Consider the phase $\psi$ of a surface wave with frequency $\omega$. We can then approximate perturbations to the phase because of lateral heterogeneity by

$$\delta \psi = \int_0^T \delta \omega_{local} dt$$

where the integration is with respect to the group travel time along the great circle path between the source and receiver. The local perturbation in eigenfrequency $\delta \omega_{local}$ is defined using the notation of Jordan (1978)

$$\delta \omega_{local}(\theta, \phi) = k \delta \omega(\theta, \phi)$$

where $\delta \omega$ is the perturbation to the phase velocity and the wave number $k$ is given by eqn [121]. Woodhouse and Dziewonski (1984) showed that the phase perturbations for both odd and even orbits may be mimicked in terms of a fictitious frequency shift $\delta \omega$ and fictitious perturbation in epicentral distance $\delta \theta$, that is

$$\delta \omega = {1 \over T} \int_0^T \delta \omega_{local} dt = \text{great circle average}$$

$$\delta \theta = \left[ {T_1 \over (1 + \frac{T}{2})} \right] (\delta \omega - \delta \omega_{local})$$

where

$$\delta \omega_{local} = {1 \over T_1} \int_0^{T_1} \delta \omega_{local} dt = \text{minor arc average}$$

and $T$ is the great circle group travel time and $T_1$ is the minor arc travel time. This result is in agreement with Jordan (1978) that the apparent frequency shift is the great circle average of $\delta \omega_{local}$, and that the apparent distance shift depends upon the difference between the great circle and minor circle averages. Along branch coupling (i.e., between modes with the same over tone number $n$) is automatically included by taking the difference between the major and minor arc frequency shifts, and thus this method also provides information on odd-degree structure in addition to even-degree structure. It gives a simple way of calculating seismograms, which incorporate the effects of phase delays along incomplete arcs, by replacing the frequency of each mode and the epicentral distance of each path by their perturbed equivalents. It should be noted that this approximation only incorporates an approximation to the phase perturbations and completely omits the consideration of amplitudes. Nevertheless, this simplified version of ray theory is a very useful one for waveform inversion and has been applied in many tomographic studies of mantle structure to both surface wave and long-period body wave data. It is a good approximation for the fundamental mode and low overtones, which constitute a major part of the long-period body wave signal.

Figure 21 illustrates the improvement in phase match after a joint inversion for the CMT source and three-dimensional Earth structure using the great-circle approximation. Performing a combined inversion for sources and structure has great advantages and prevents trade-off between the recovered structure after tomographic inversion and the use of a simplified (i.e., 1D) model for the source inversion. Performing the CMT inversion jointly with a structure inversion leads to a significant improvement in phase match between data and synthetics (Figure 21). Procedures for performing a combined inversion are outlined in Woodhouse and Dziewonski (1989) and Valentine and Woodhouse (2010).

The shortcoming of the great circle approximation is that it is not very accurate for direct body wave phases, since it predicts that the observed seismogram depends only upon the horizontally averaged structure. The great circle, or path-average approximation, was extended by Li and Tanimoto (1993) and Li and Romanowicz (1995) to include across branch-coupling (i.e., between modes with different over tone number $n$), which asymptotically corrects the body wave phases. In practise, nonlinear asymptotic coupling theory splits the synthetic seismogram calculation into two parts. The first part is just the great circle approximation, where the frequency and epicentral distance are replaced by their perturbed equivalents, and then a second part is added which represent the cross-branch mode coupling effects, see Romanowicz et al. (2008) for details. This method has been successfully applied to whole mantle tomography using waveforms.

1.03.9 Concluding Discussion

The normal mode formalism provides a well developed theoretical framework for the calculation of theoretical seismograms in both spherically symmetric and three-dimensional Earth models. For spherically symmetric models the ability to simply and quickly calculate complete theoretical seismograms plays an important role in the formulation and solution of many seismological problems involving both surface waves and long-period body waves. In the three-dimensional case,
the theory of mode coupling is too cumbersome to be applied in full, but it enables a number of useful approximations to be developed and tested. The increasing capacity in high performance computing means that it becomes possible to develop and test increasingly more complete implementations of the fully coupled theory. Progress on fully numerical solutions for seismic wave fields in realistic three-dimensional spherical models (Komatitsch and Tromp, 2002a,b), while it is providing a new and invaluable tool in many areas of global seismology, has not yet made it possible to calculate accurate very long-period spectra. In part this is because a way has not (yet?) been found to fully implement self-gravitation in the spectral element method, and in part because the small time step needed in finite difference and spectral element calculations leads to very long execution times; also, there are very stringent limits on the tolerable amount of numerical dispersion in the solution.

Long-period modal spectra constitute a rich source of information on long wavelength heterogeneity, studies to date, we believe, having only scratched the surface. To realize the potential of this information will require large-scale coupling calculations or, possibly, other methods for calculating very long-period wave fields yet to be developed. This will make it possible to bring modal spectral data increasingly to bear on furthering our understanding of the Earth’s three-dimensional structure.
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